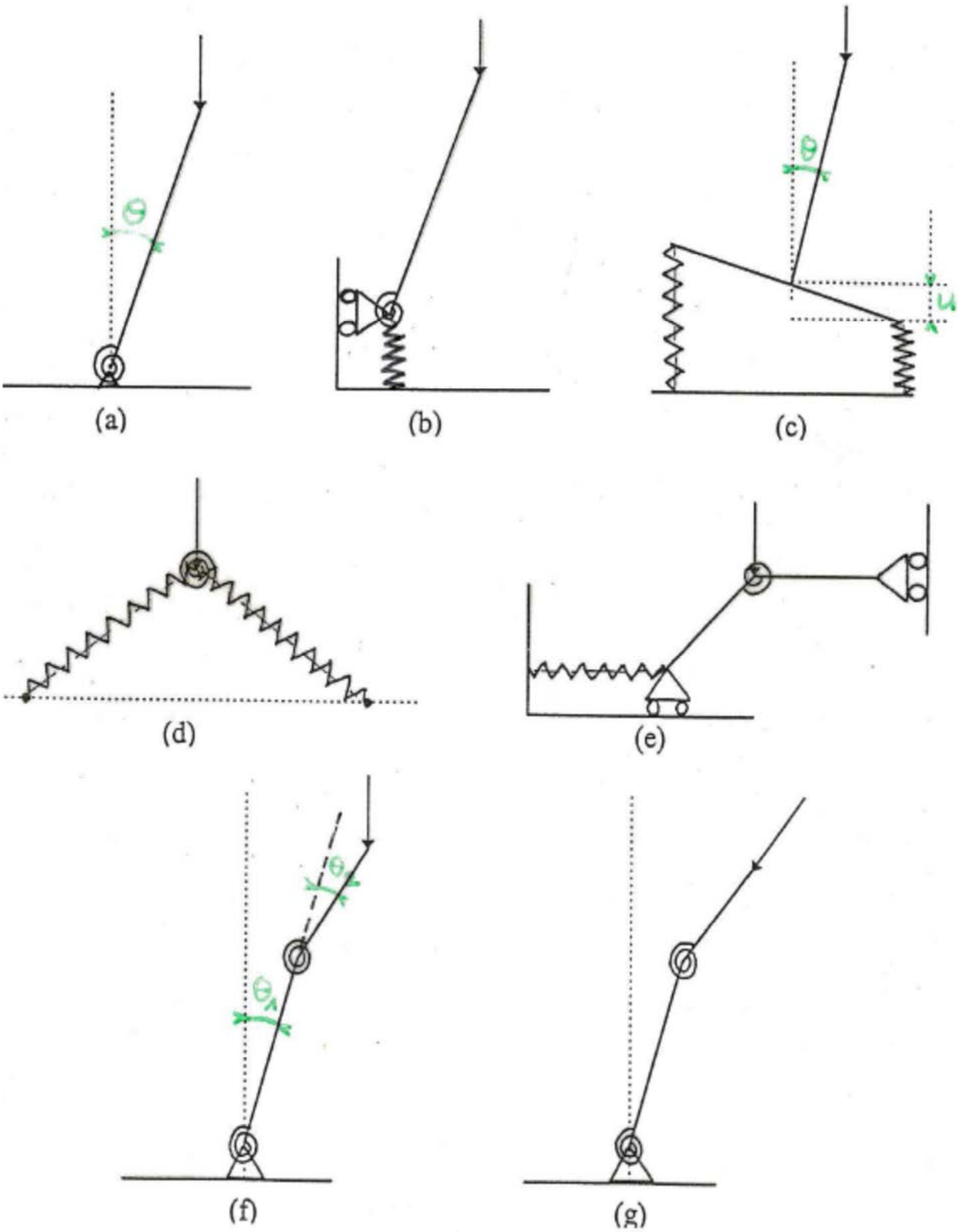
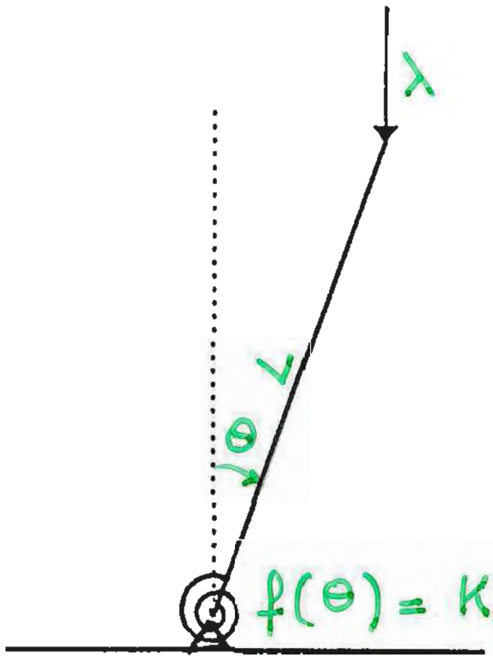
SIMPLE MODELS AND STRUCTURES

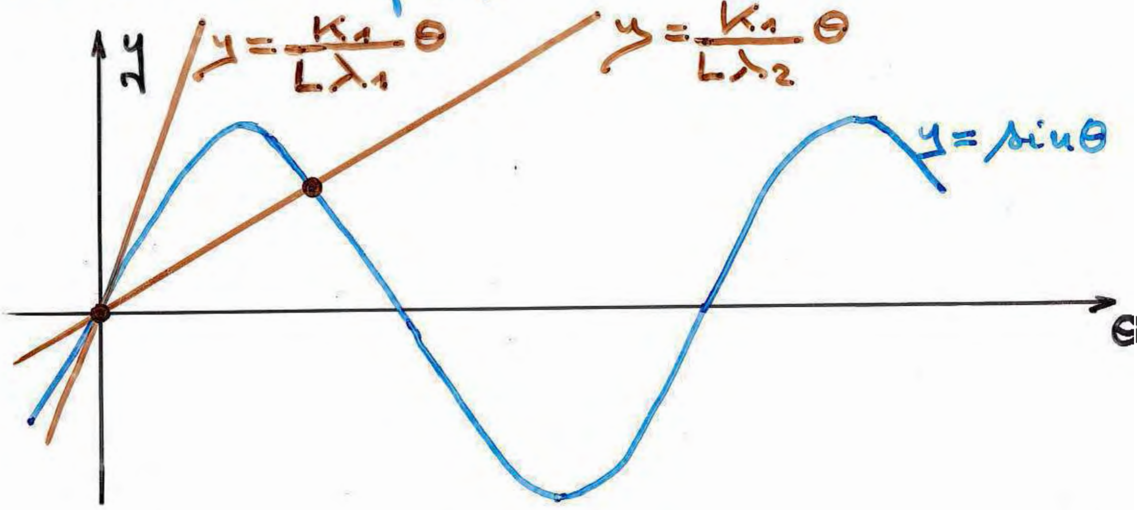




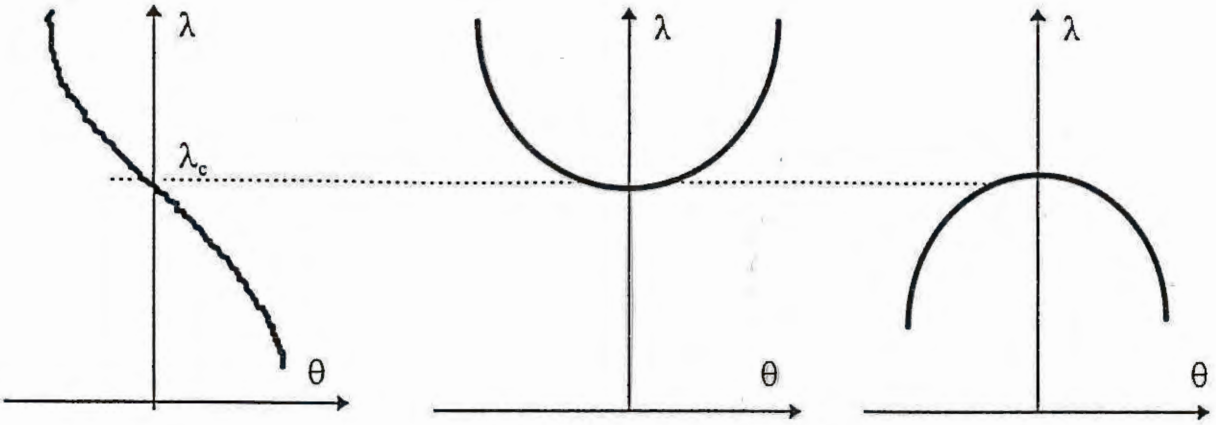
$$f(\theta) = K_1\theta + K_2\theta^2 + K_3\theta^3 + \dots$$

$$K_1 > 0$$

$$\lambda L \sin\theta = f(\theta)$$



$$\begin{cases} \theta = 0, \quad \forall \lambda \in R^+ \\ \lambda = \dfrac{f(\theta)}{L \sin\theta} \end{cases}$$



(a) $K_2 < 0$ (b) $K_2 = 0, \lambda_1 = 0, \lambda_2 > 0$ (c) $K_2 = 0, \lambda_1 = 0, \lambda_2 < 0$

$$\lambda = \lambda_c + \lambda_1 \theta + \lambda_2 \theta^2 + \cdots$$

$$\begin{cases} \lambda_c = \dfrac{K_1}{L} \\ \lambda_1 = \lambda_c \dfrac{K_2}{K_1} \\ \lambda_2 = \lambda_c \left[\dfrac{K_3}{K_1} + \dfrac{1}{6} \right] \\ \vdots \end{cases}$$

THE COMPRESSED BEAM

Kinematics :

$$\begin{cases} \varepsilon(x) = u' + \dfrac{(v')^2}{2} , & (\cdot)' \equiv \dfrac{d}{dx} \\ \kappa(x) = -v'' \end{cases}$$

$$\hookrightarrow \gamma(x,y) = u'(x) + \frac{(v'(x))^2}{2} - y\, v''(x)$$

Equilibrium :

$$\begin{cases} N' = 0 \\ M'' + N\kappa = 0 \end{cases}$$

Constitutive law :

$$\begin{cases} N = ES\left(u' + \dfrac{(v')^2}{2}\right) \\ M = EI\, v'' \end{cases}$$

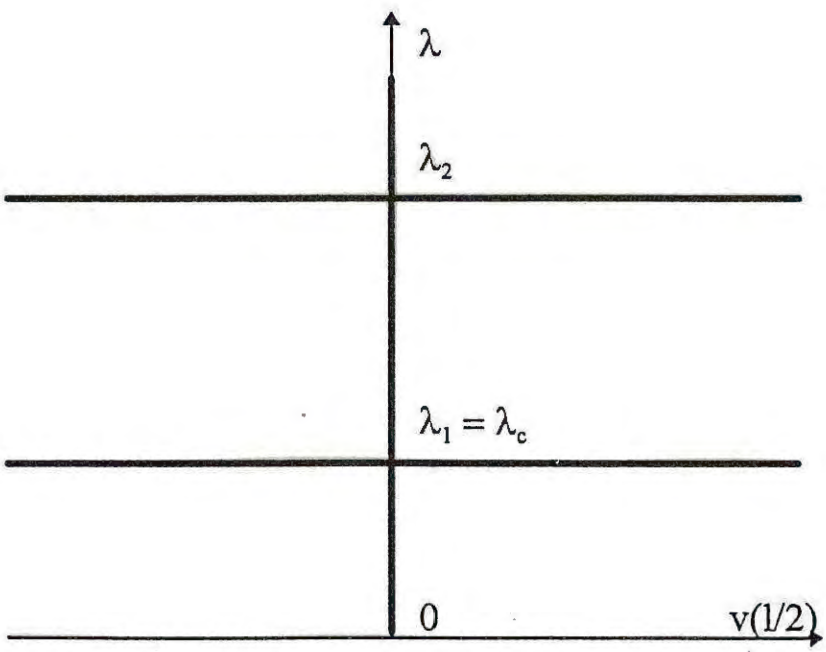
$$\hookrightarrow \boxed{\begin{array}{l} N = C^{ste} = -\lambda \\ (EI\, v'')'' + \lambda v'' = 0 \\ + \text{B.C.} \end{array}}$$

* Uniform cross-section
* S.S. Boundary Conditions

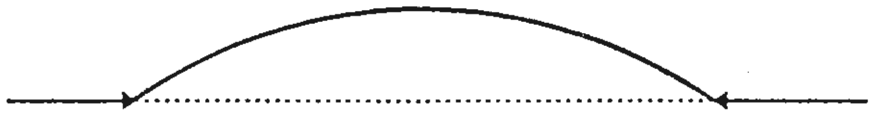
$$\begin{cases} EI\,v'' + \lambda v = 0 \\ v(0) = v(\ell) = 0 \end{cases}$$

$\Rightarrow \quad \lambda_n = \dfrac{n^2 \pi^2 EI}{\ell^2}$

$\Rightarrow$ Bifurcation diagram:

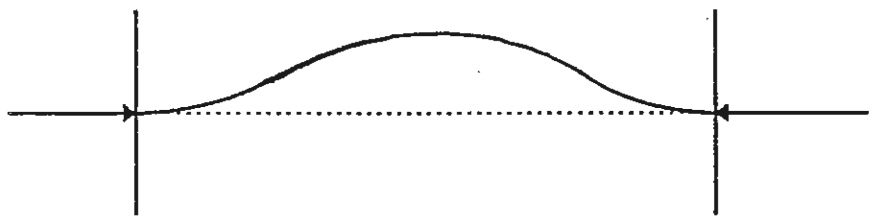


- Extrémités simplement appuyées:



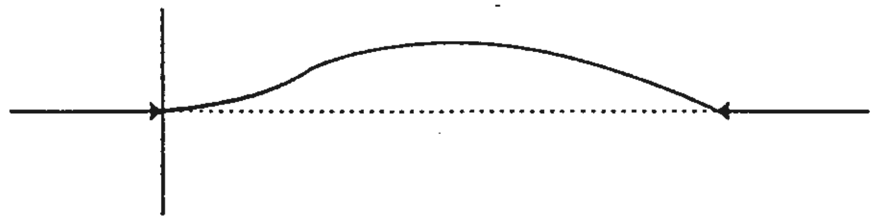
$$\lambda_c = \frac{\pi^2 EI}{\ell^2}$$

- Extrémités encastrées:



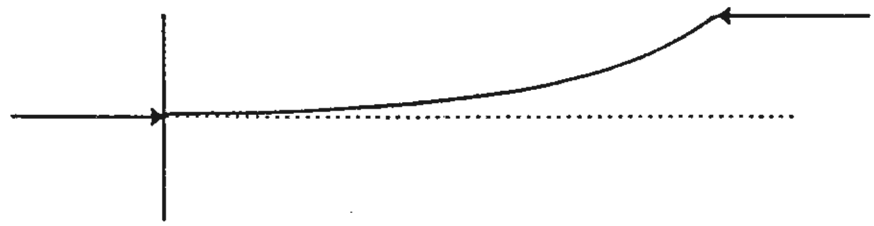
$$\lambda_c = \frac{4\pi^2 EI}{\ell^2}$$

- Appui simple à une extrémité, encastrement à l'autre:

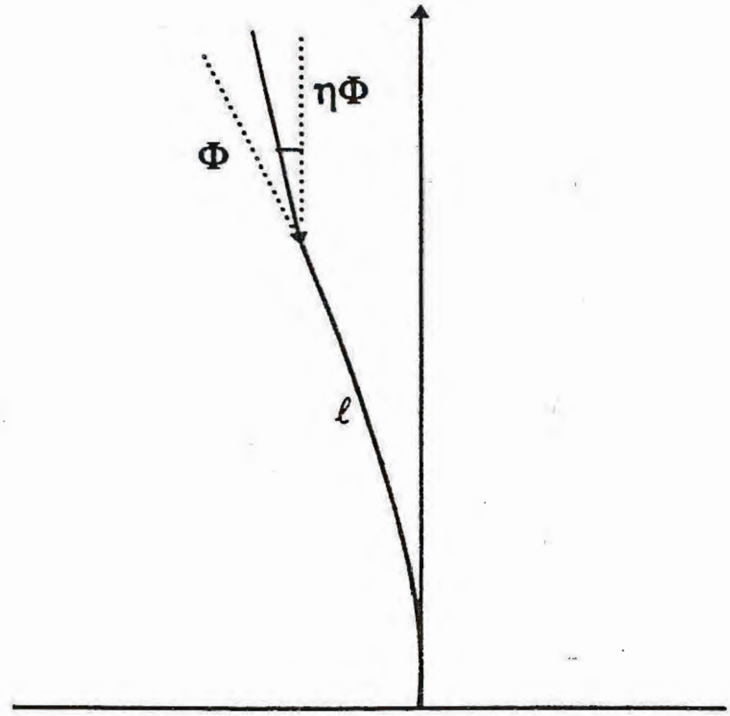


$$\lambda_c = 2.04\frac{\pi^2 EI}{\ell^2}$$

- Une extrémité encastrée, l'autre libre:



$$\lambda_c = \frac{\pi^2 EI}{4\ell^2}$$



$$EI\, v_{,xxxx} + \lambda\, v_{,xx} + m\, v_{,tt} = 0$$

$$v(x,t) = e^{i\omega t} f(x) \Rightarrow \begin{cases} EI\, f^{(4)} + \lambda f'' - m\omega^2 f = 0 \\ f(0) = f'(0) = f''(L) = 0 \\ f'''(L) - (\eta - 1)\dfrac{\lambda}{EI} f'(L) = 0 \end{cases}$$

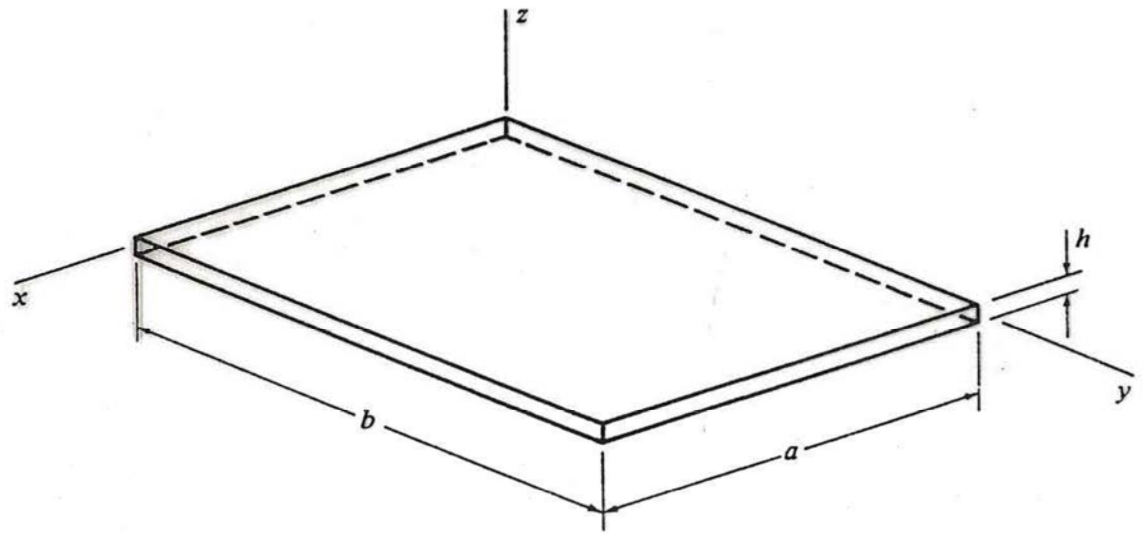
1°/ $\underline{\omega = 0}$:

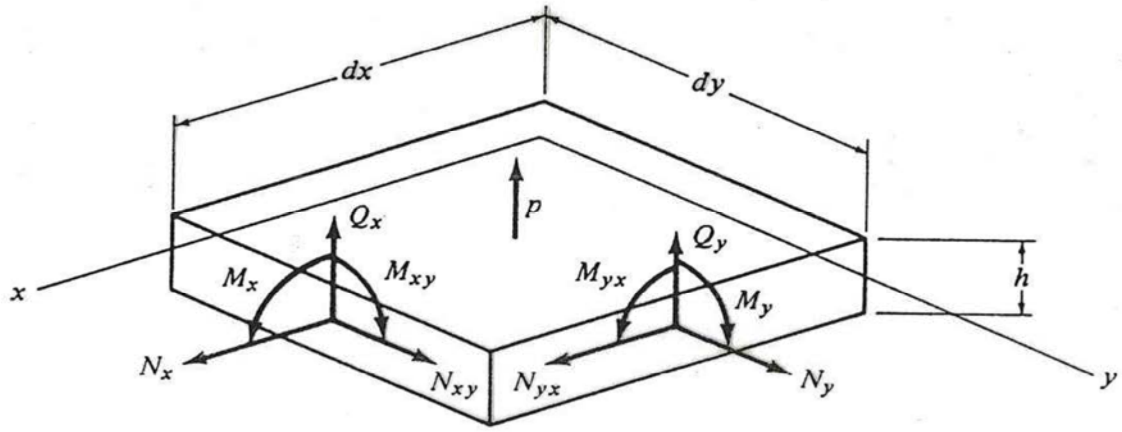
$$\boxed{\lambda_c = \frac{\pi^2 EI}{L^2} \quad \frac{1}{\pi^2}\left[\arccos \frac{\eta}{\eta - 1}\right]^2}$$

$\Rightarrow \eta \leq 0.5$!

2°/ $\underline{\eta > 0.5}$ $\Longrightarrow$ $\omega^2 = 1.24 \dfrac{\pi^2 EI}{m L^4}$ à $\lambda = 20.05 \dfrac{EI}{L^2}$

THE VON KARMAN PLATE





Kirchhoff - Love kinematics:

$$\begin{cases} \bar{u} = u + z\,\beta_x \\ \bar{v} = v + z\,\beta_y \\ \bar{w} = w \end{cases} \quad \text{where} \quad \begin{cases} \beta_x = -w_{,x} \\ \beta_y = -w_{,y} \end{cases}$$

u, v, w : fcts of (x, y) only.

* Small strains, finite displacements

$$\begin{cases} \varepsilon_x = u_{,x} + \frac{1}{2}(w_{,x})^2 \\ \varepsilon_y = v_{,y} + \frac{1}{2}(w_{,y})^2 \\ \gamma_{xy} = u_{,y} + v_{,x} + w_{,x}\, w_{,y} \end{cases} \qquad \begin{cases} \kappa_x = -w_{,xx} \\ \kappa_y = -w_{,yy} \\ \kappa_{xy} = -w_{,xy} \end{cases}$$

* Constitutive law:

$$\begin{cases} N_x = C\left(\varepsilon_x + \nu \varepsilon_y\right) \\ N_y = C\left(\varepsilon_y + \nu \varepsilon_x\right) \\ N_{xy} = C\,\frac{1-\nu}{2}\,\gamma_{xy} \end{cases} \qquad \begin{cases} M_x = D\left(\kappa_x + \nu \kappa_y\right) \\ M_y = D\left(\kappa_y + \nu \kappa_x\right) \\ M_{xy} = D(1-\nu)\,\kappa_{xy} \end{cases}$$

with:

$$C \equiv \frac{E h}{1-\nu^2} \; ; \quad D \equiv \frac{E h^3}{12(1-\nu^2)}$$

* Equilibrium : Von Kármán equations

$$\boxed{\begin{aligned} & N_{x,x} + N_{xy,y} = 0 \\ & N_{xy,x} + N_{y,y} = 0 \\ & D\,\nabla^4 w - \left(N_x\, w_{,xx} + 2 N_{xy}\, w_{,xy} + N_y\, w_{,yy}\right) = p \end{aligned}}$$

REMARK:

* Let's introduce a "stress function" $f(x,y)$ by:

$$\begin{cases} N_x = f_{,yy} \\ N_y = f_{,xx} \\ N_{xy} = -f_{,xy} \end{cases}$$

* and a braket (Monge-Ampère form):

$$[f,g] \stackrel{\text{def}}{=} f_{,xx}\,g_{,yy} + f_{,yy}\,g_{,xx} - 2 f_{,xy}\,g_{xy}$$

* Then the set of von Kármán equations is changed into:

$$\begin{aligned} \nabla^4 f &= -Eh[w,w] \\ D\nabla^4 w &= [f,w] + p \end{aligned}$$

$+$

B.C.

Assume for a given load p, there exists a solution (u_0, v_0, w_0) corresponding to $N_{x0}, N_{y0}, \dots$

* "Adjacent" equilibrium :

$\exists ? \ (\hat{u}, \hat{v}, \hat{w}) \not\equiv 0$ &

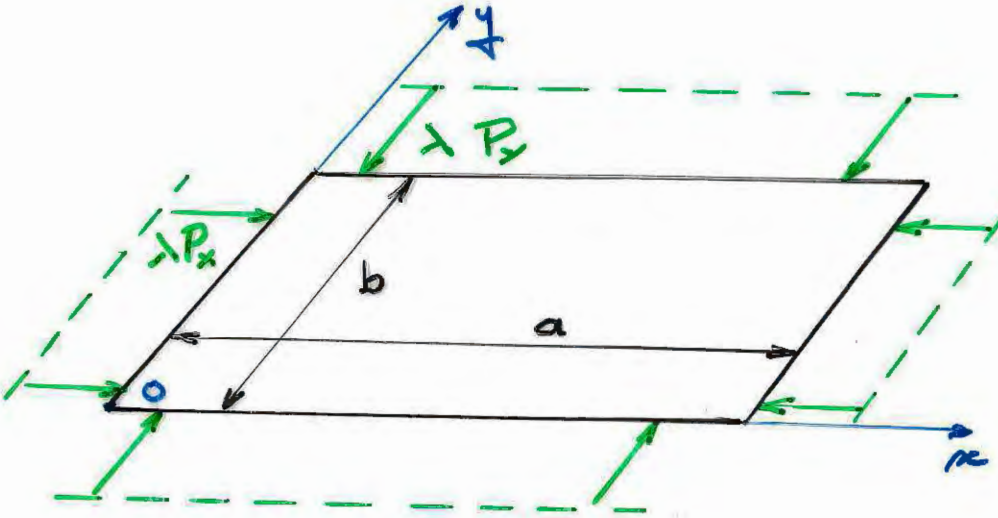
$u = u_0 + \hat{u}$, $v = v_0 + \hat{v}$, $w = w_0 + \hat{w}$

is also a solution for the same p

Then:

$$\boxed{\begin{aligned} &\hat{N}_{x,x} + \hat{N}_{xy,y} = 0 \\ &\hat{N}_{xy,x} + \hat{N}_{y,y} = 0 \\ &D\nabla^4 \hat{w} - \left(N_{x0}\hat{w}_{,xx} + 2N_{xy0}\hat{w}_{,xy} + N_{y0}\hat{w}_{,yy}\right) = 0 \end{aligned}}$$

- Homogeneous
- Uncoupled
- Linear.



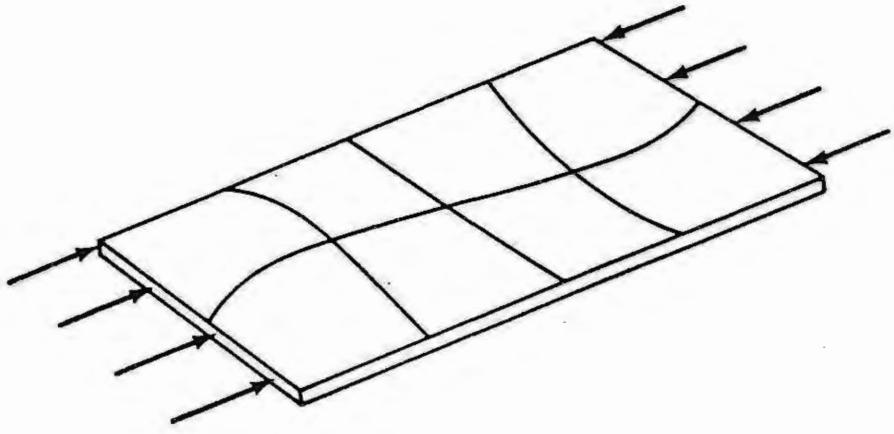
$$\begin{cases} N_{x0} = -\lambda P_x \; ; \; N_{y0} = -\lambda P_y ; \; N_{xy0} = 0 ; \\ D \nabla^4 w + \lambda P_x w_{,xx} + \lambda P_y w_{,yy} = 0 \\ w = w_{,xx} = 0 \quad \text{en} \quad x = 0 \text{ et } x = a \\ w = w_{,yy} = 0 \quad \text{en} \quad y = 0 \text{ et } y = b \end{cases}$$

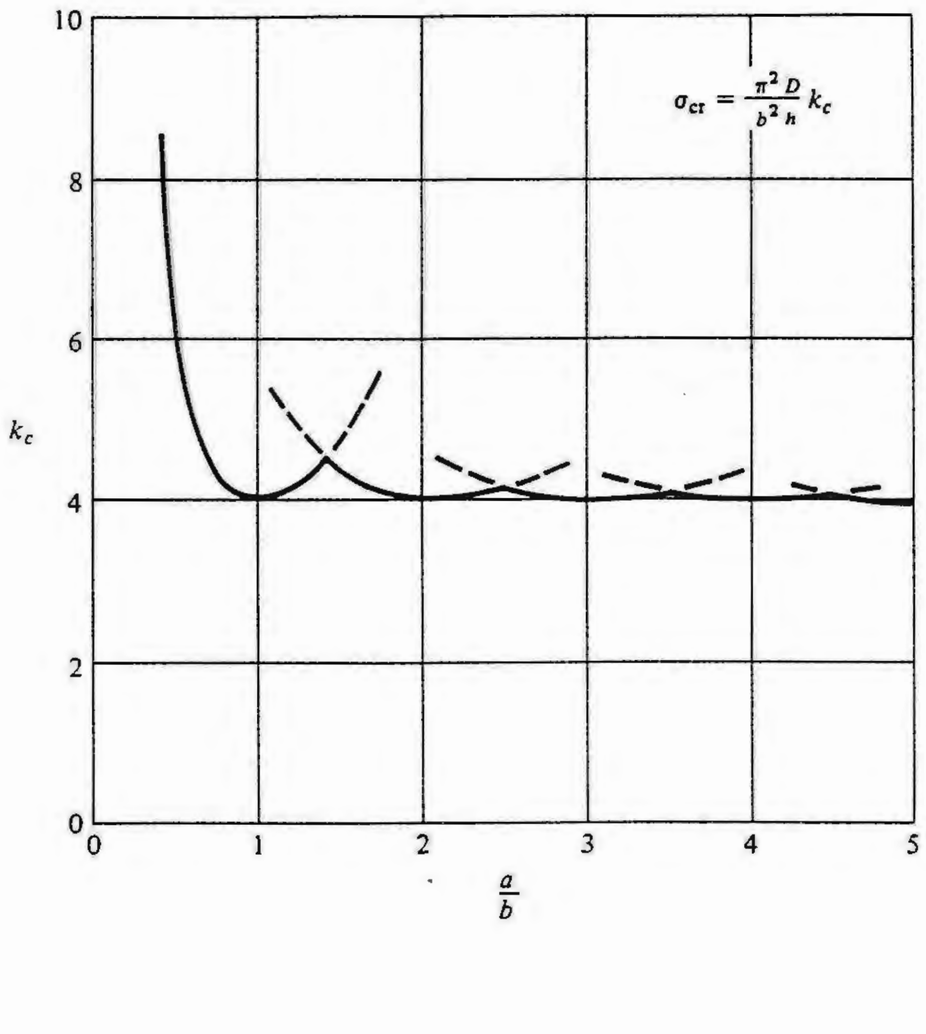
* $w(x,y) = \sin \dfrac{m\pi x}{a} \sin \dfrac{n\pi y}{b}$

$$\boxed{\lambda_c = k \pi^2 D \;, \quad k = \frac{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}{\left(\frac{m}{a}\right)^2 P_x + \left(\frac{n}{b}\right)^2 P_y}}$$

* $P_y = 0 \Longrightarrow n = 1$

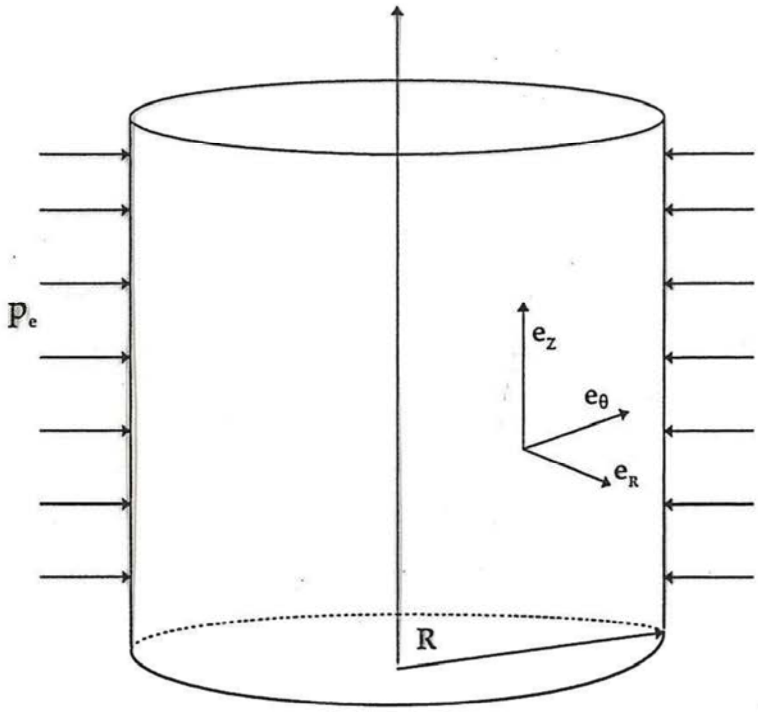
$$\boxed{\lambda_c = k \frac{\pi^2 D}{b^2} \;, \quad k = \left(\frac{mb}{a} + \frac{a}{mb}\right)^2 \frac{1}{P_x}}$$





10
8
6
4
2
0
k_c
$\sigma_{cr} = \frac{\pi^2 D}{b^2 h} k_c$
0
1
2
3
4
5
$\frac{a}{b}$

CYLINDRICAL SHELLS



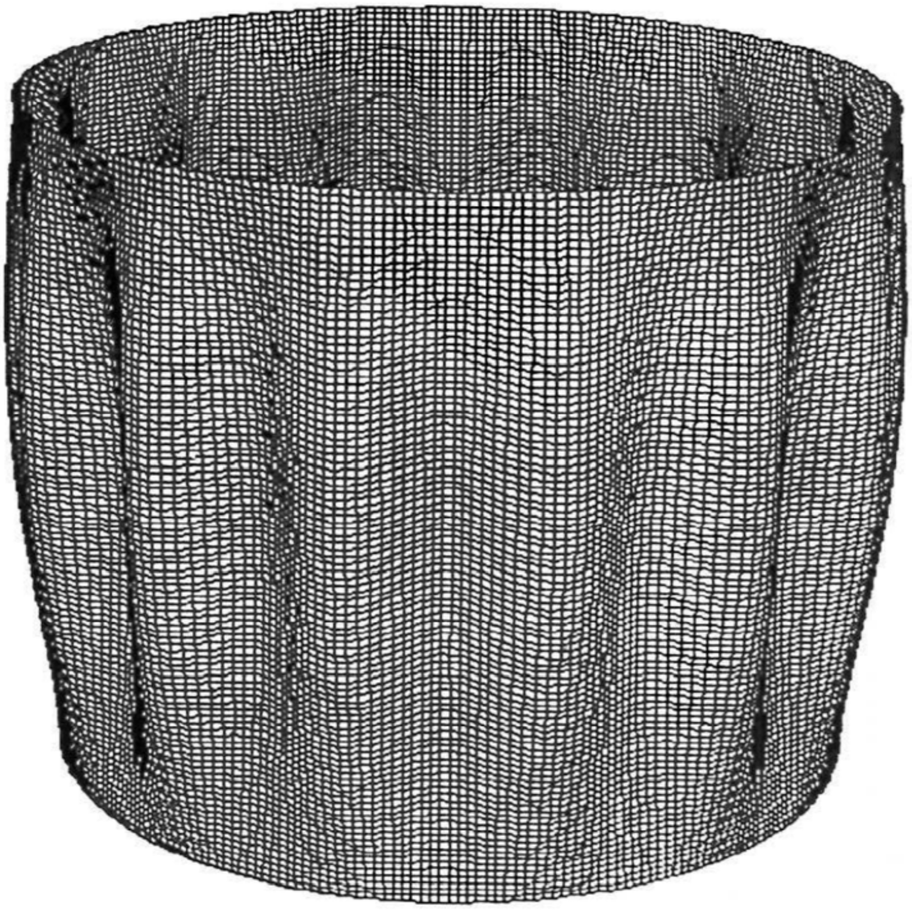
$$\begin{cases} D\nabla^4 w + R f_{,zz} - \left[f_{,\theta\theta} w_{,zz} - 2 f_{,z\theta} w_{,z\theta} + f_{,zz} w_{,\theta\theta}\right] = p \\ \nabla^4 f - \dfrac{Eh}{R^4}\left[w_{,z\theta}^2 - w_{,zz} w_{,\theta\theta} + R\, w_{,zz}\right] = 0 \end{cases}$$

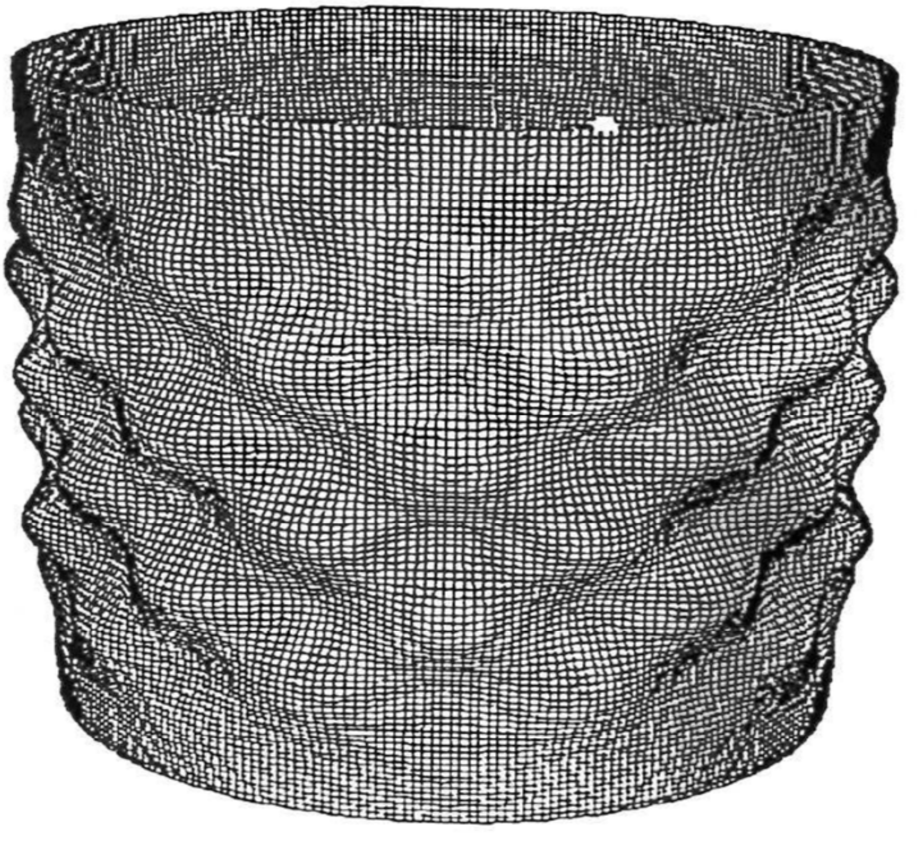
Perturbation autour de $N_\theta = -p_e R$:

$$\begin{cases} D\nabla^8 w + \dfrac{1-\nu^2}{R^2} C\, w^{(4)}_{,xxxx} + \dfrac{1}{R} p_e \nabla^4 w_{,\theta\theta} = 0 \\ + \text{C.L.} \end{cases}$$

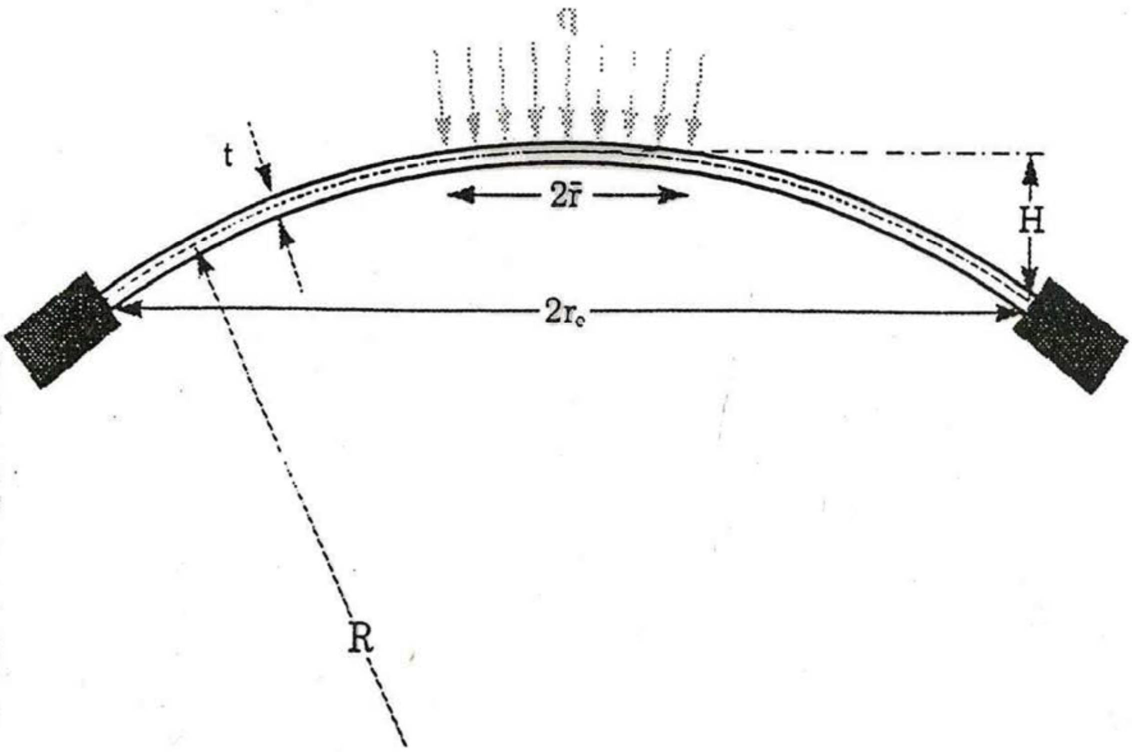
$\hookrightarrow$ $w = A \sin \bar{m} x \cdot \sin n\theta$, $\bar{m} = \dfrac{m\pi R}{L}$

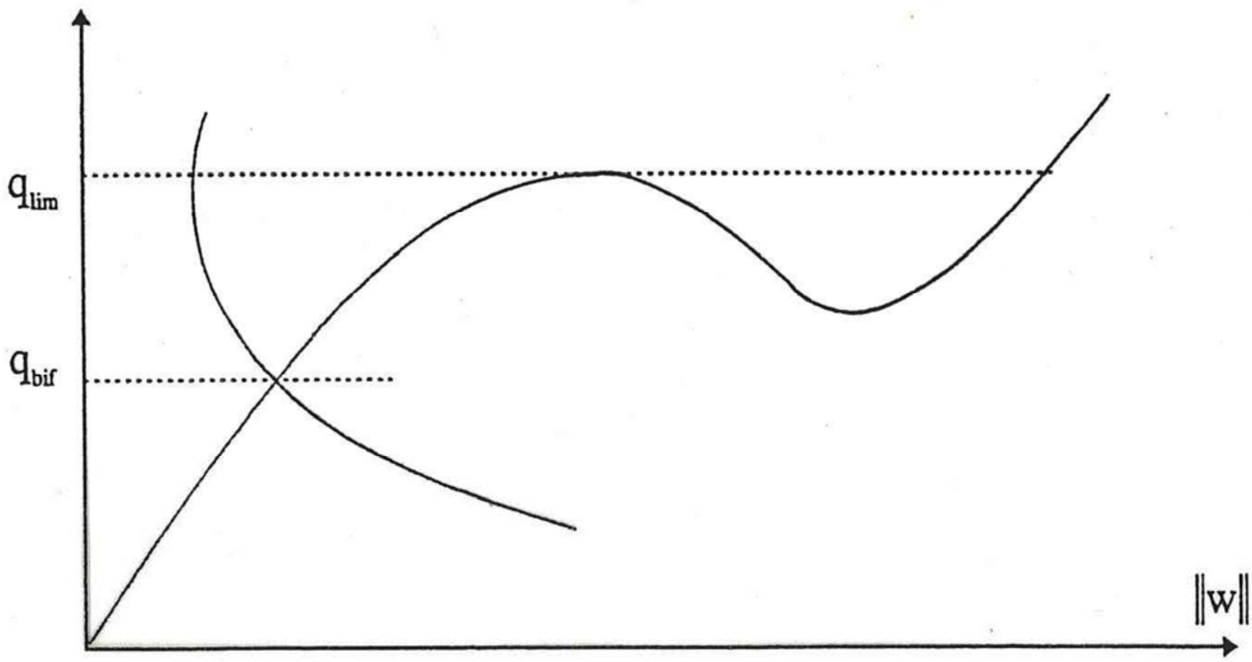
$\hookrightarrow$ $$\boxed{p_e R = \frac{(\bar{m}^2 + n^2)^2}{n^2}\frac{D}{R} + \frac{\bar{m}^4}{n^2(\bar{m}^2 + n^2)^2}(1-\nu^2)C}$$

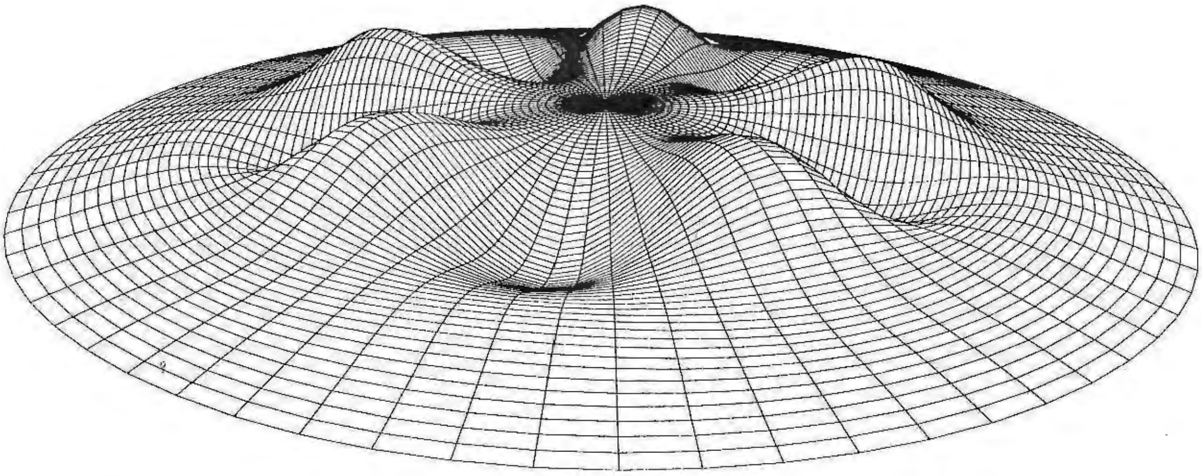




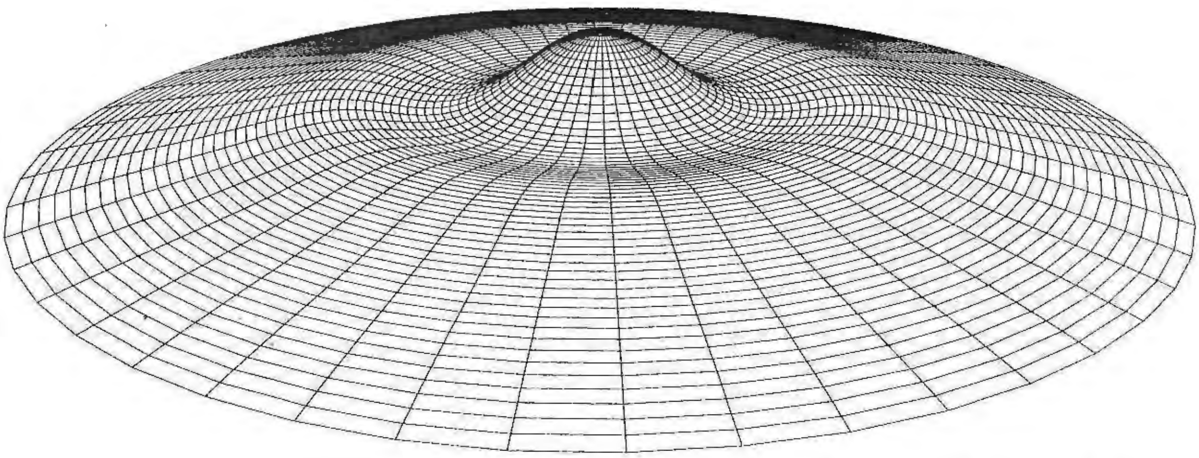
SPHERICAL CAP





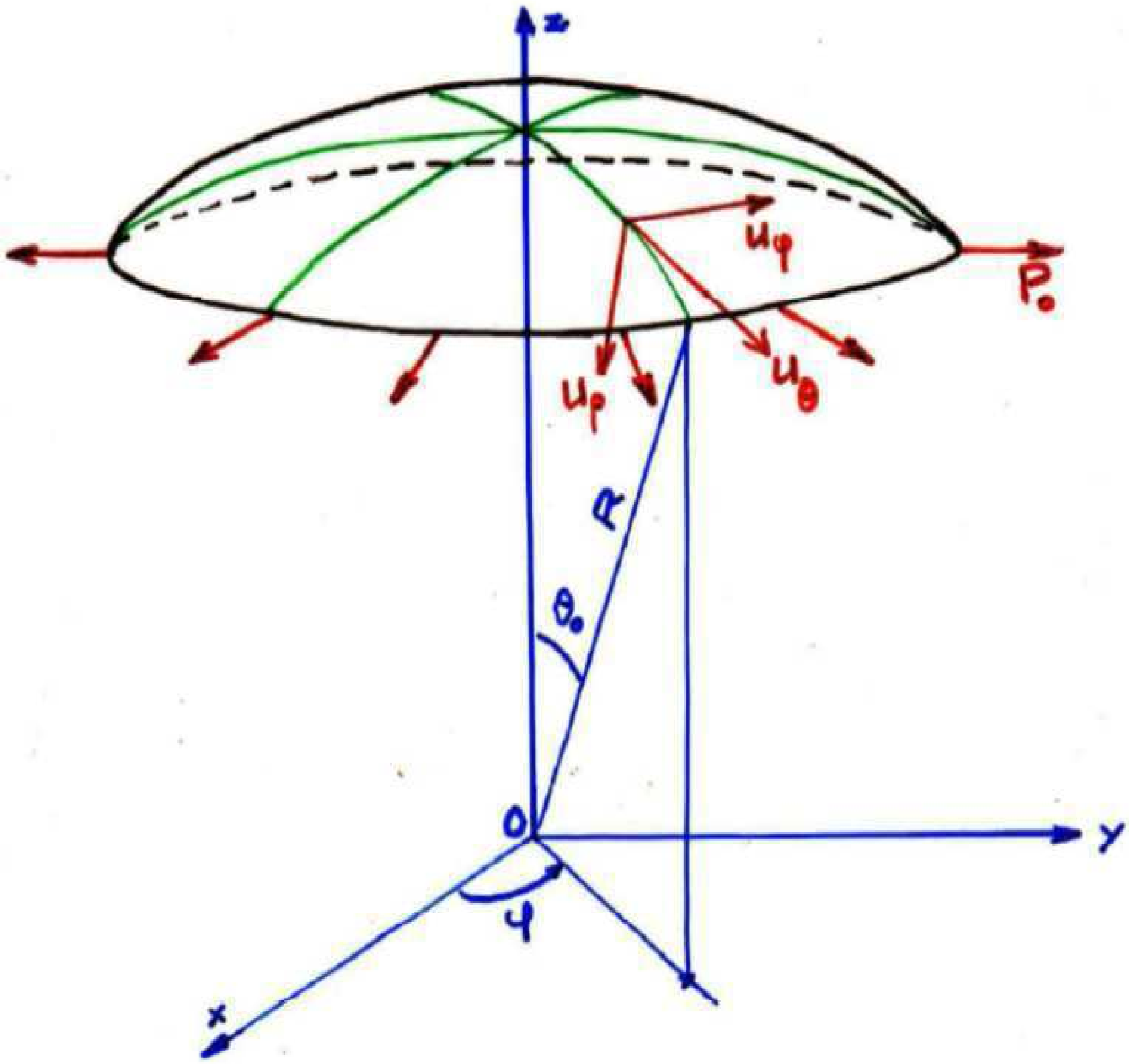


ALU - lamda = 9 - lamda barre = 8.4 - flambage en mode 5



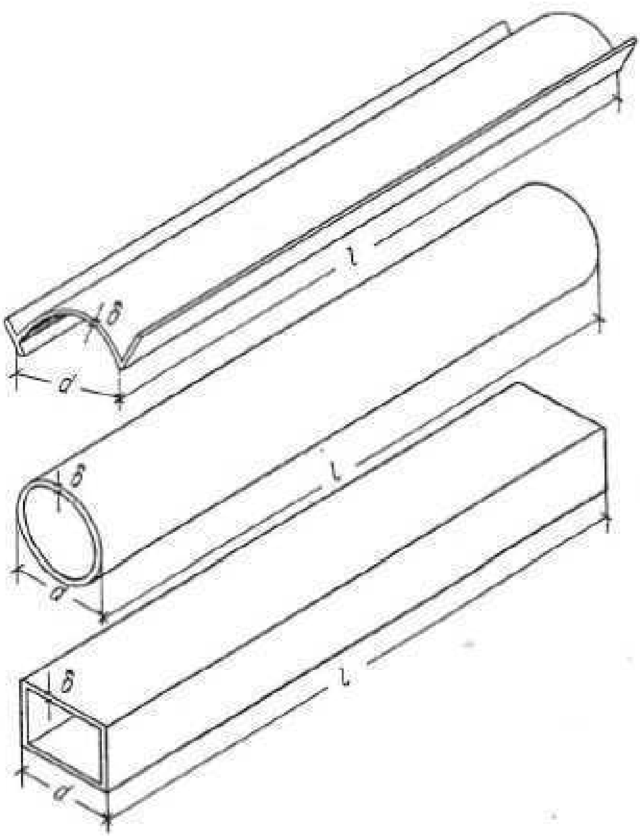
ALU - lamda = 9 - lamda barre = 6.5 - flambage en mode 0

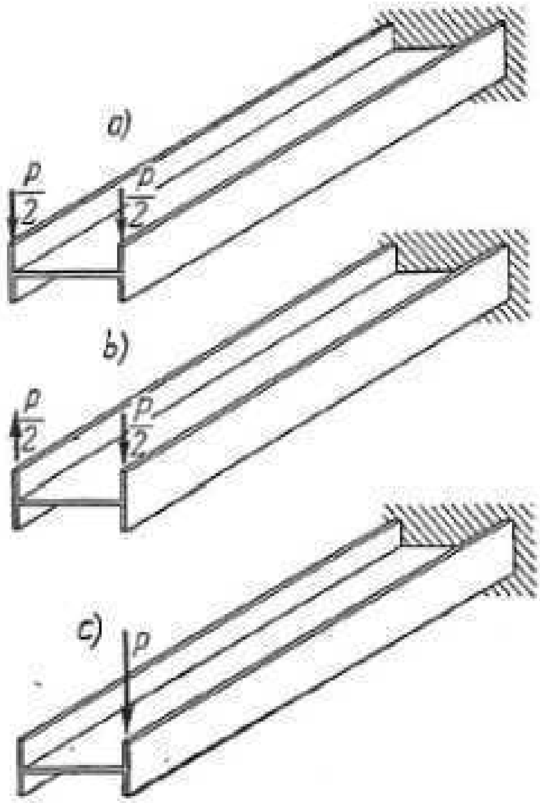
MULTI PARAMETER PROBLEMS

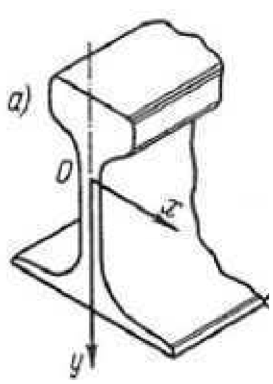


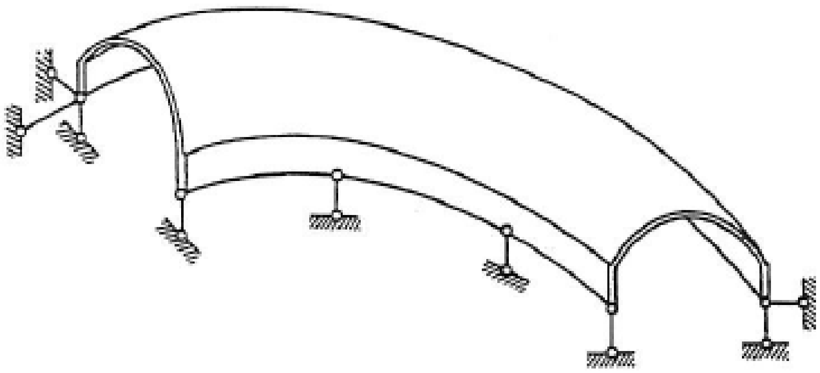
Thin-walled structures

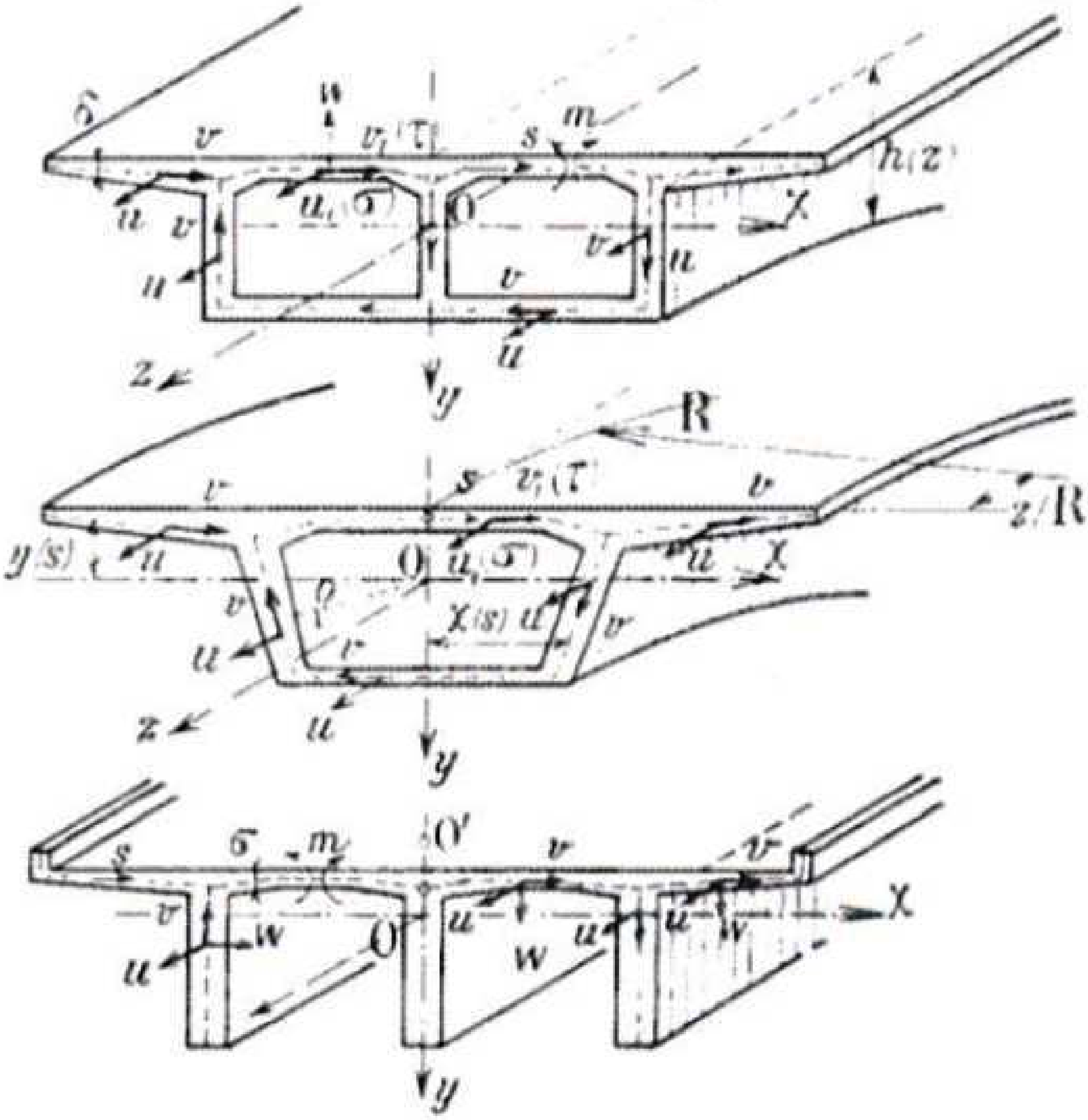
Open or closed cross sections :











w
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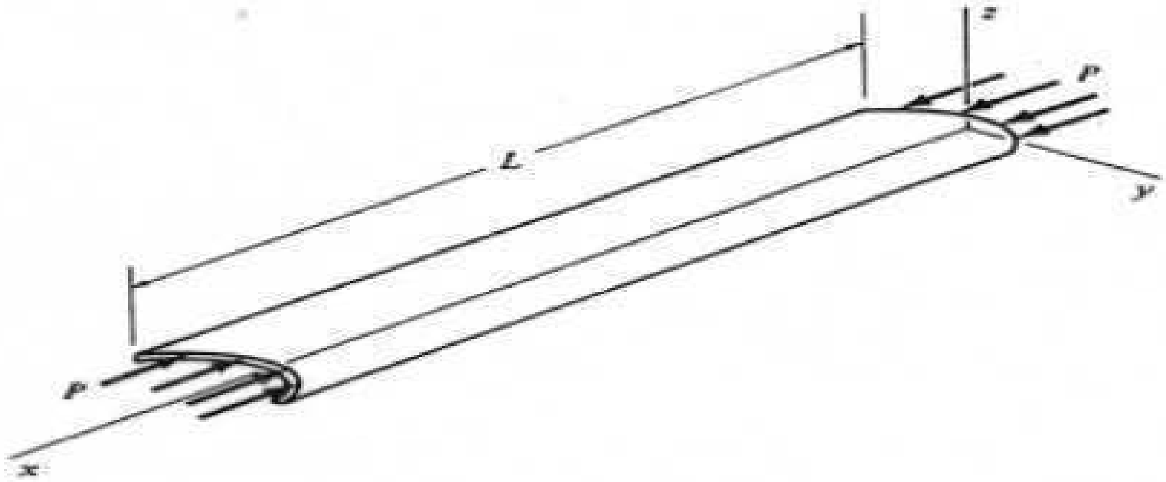
X(s)
O'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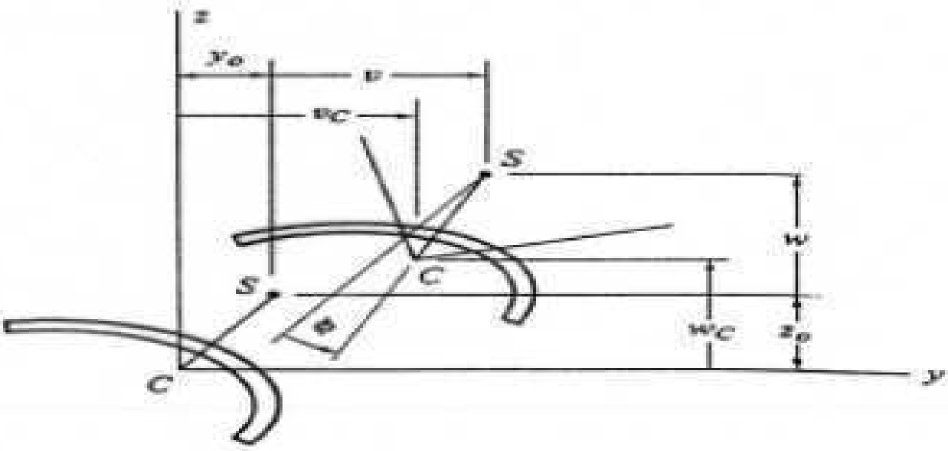
z
y_o
v
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S
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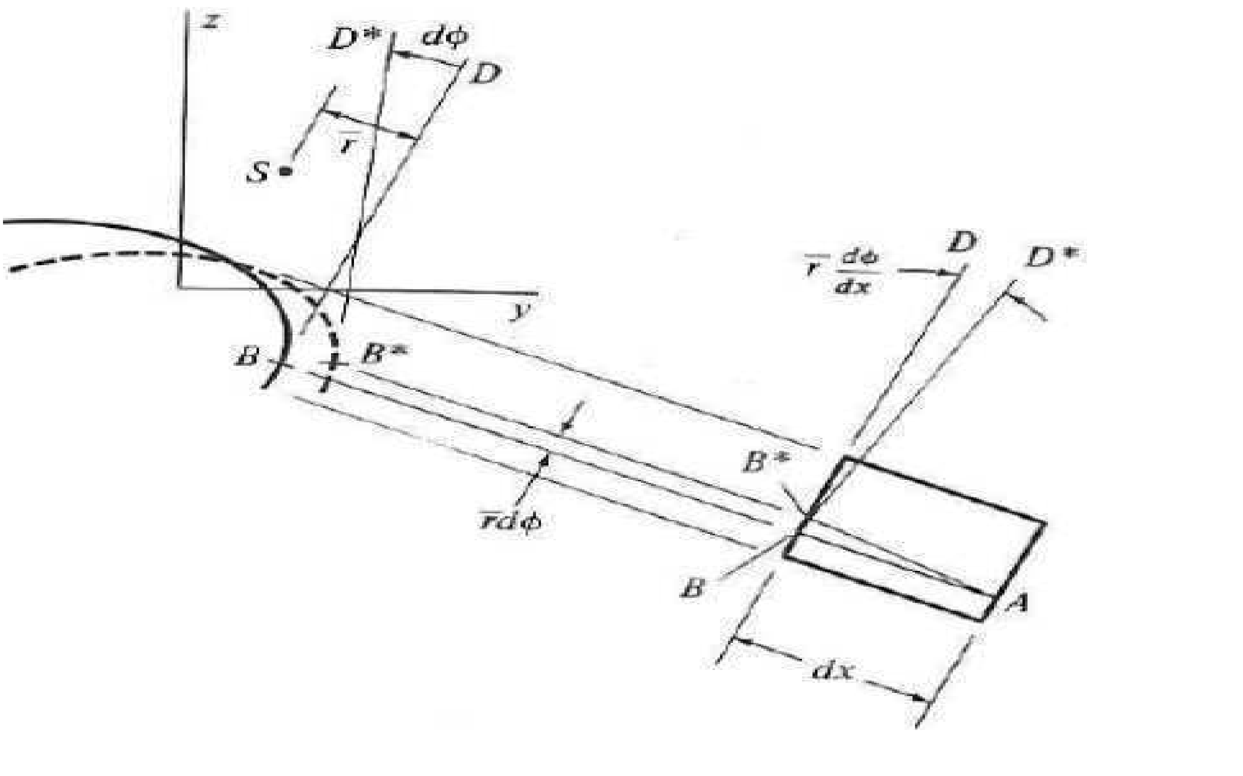
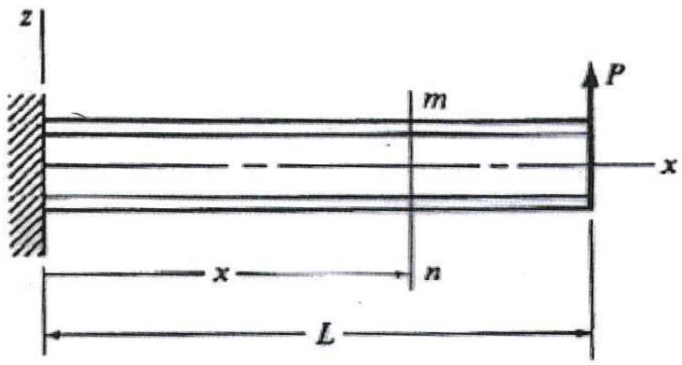
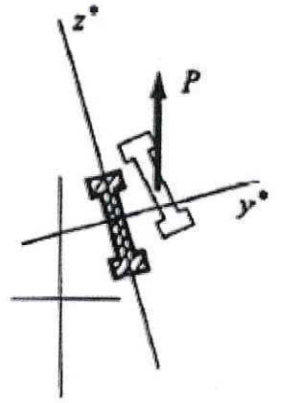
w
w_C
z_o
S
C
y


z
D^*
$d\phi$
D
S
$\overline{r}$
y
B
B^*
$\overline{r}\,\frac{d\phi}{dx}$
D
D^*
B^*
$\overline{r}d\phi$
B
A
dx





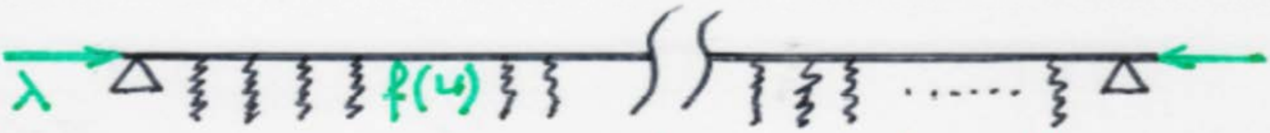
$$E C_w \phi'''' - GJ\,\phi'' - \frac{P^2}{E I_{33}}(L-x)^2\,\phi = 0$$

$$\xi = L - x, \quad k = \frac{P^2}{E I_{33}} GJ$$

$$\longrightarrow \quad \frac{d^2\phi}{d\xi^2} + k^2 \xi^2 \phi = 0$$

$$k = \frac{4.013}{L^2} \quad \text{or} \quad P_{cr} = 4.01\frac{(E I_{33} GJ)^{1/2}}{L^2}$$

Very long structures



$$f(u) = k_1 u + k_3 u^3 + \text{---}$$

$$u'''' + \lambda u'' + (\operatorname{sgn} k_3) u^3 + u = 0$$

$$u'''' + \lambda u'' + u = 0$$

$$\lambda_c = 2$$

$$\varepsilon = \lambda - 2 \quad , \quad X = \varepsilon^{1/2} x$$

$$u = \varepsilon^{1/2} u_1 + \varepsilon u_2 + \text{---}$$

$$u_1 = A_1(X)\, e^{ix} + c.c.$$

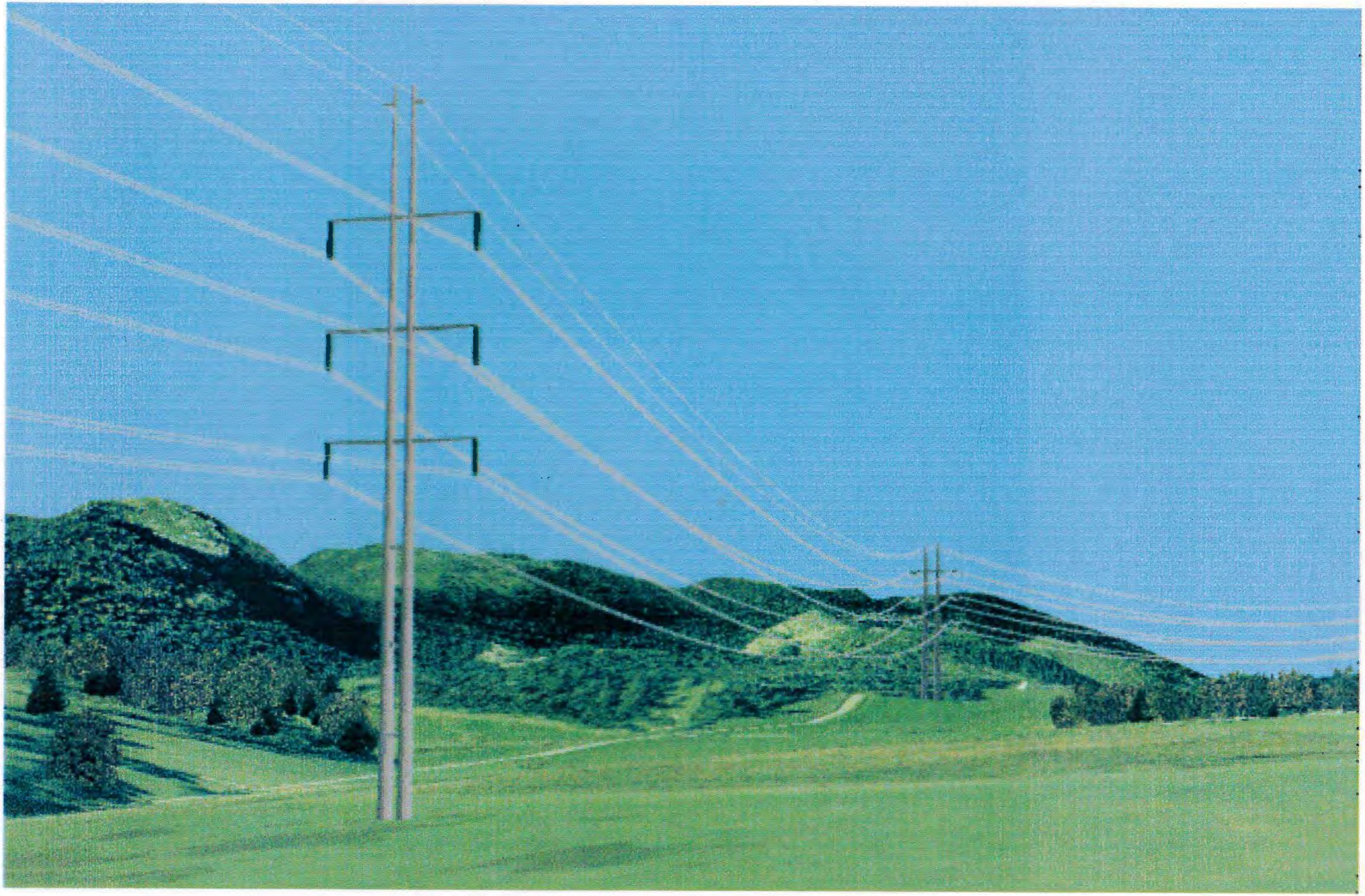
$$A_1(X) = r(X)\, e^{i\theta(X)}$$

$$4A_1'' + A_1 - A_1 |A_1|^2 = 0$$

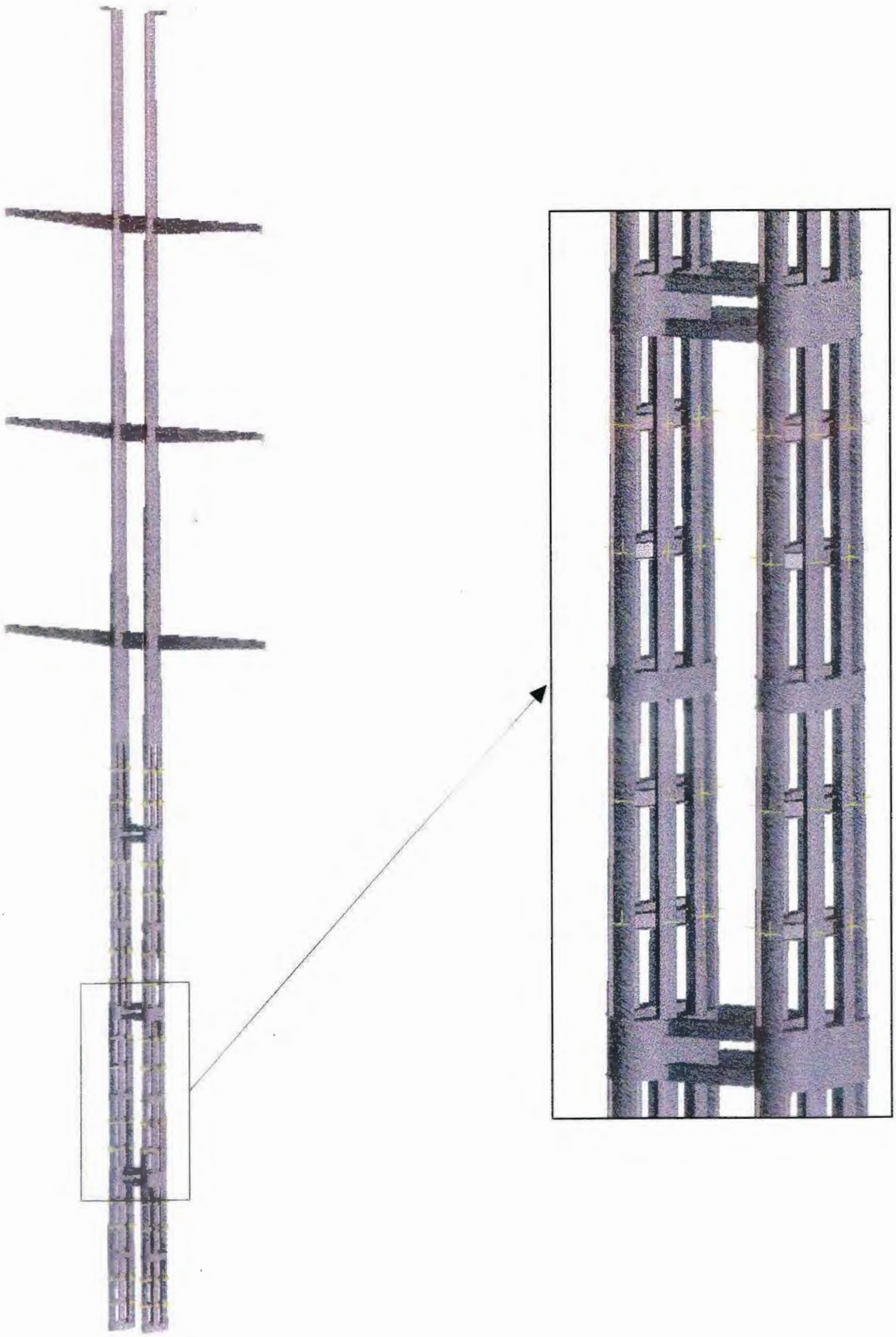
$$a(X) = \varepsilon^{1/2} A_1(X)$$

* Amplitude modulation

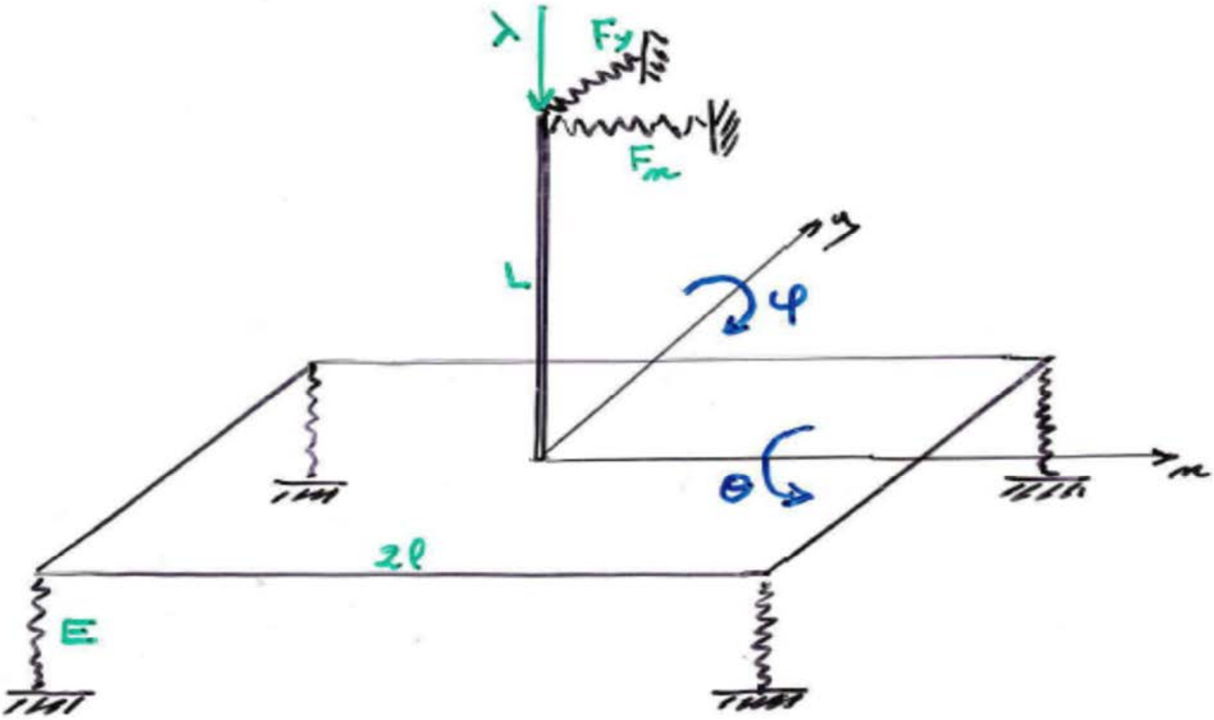
* Localization



Pylônes MIMRAM

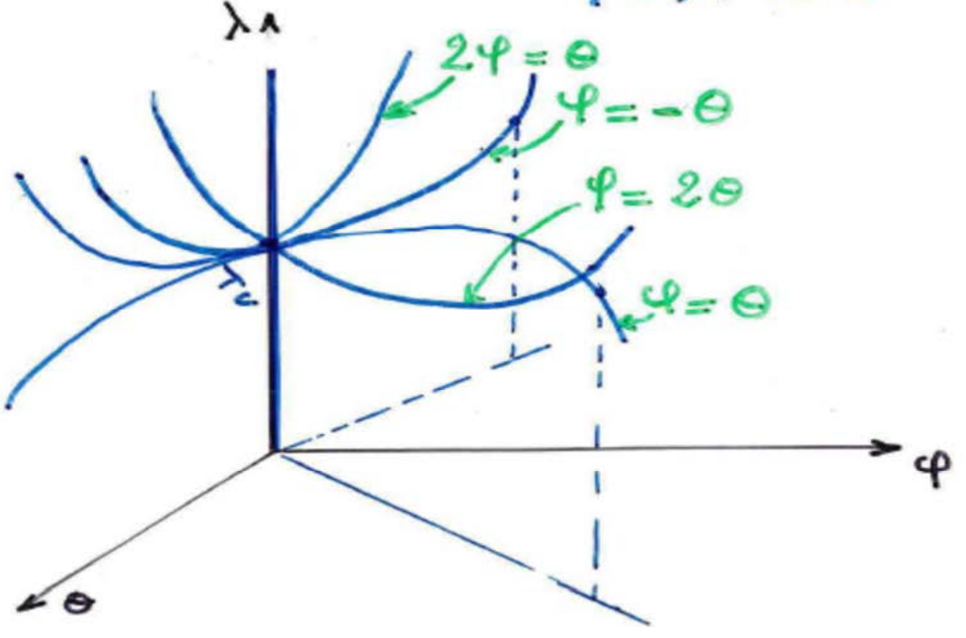


A simple model

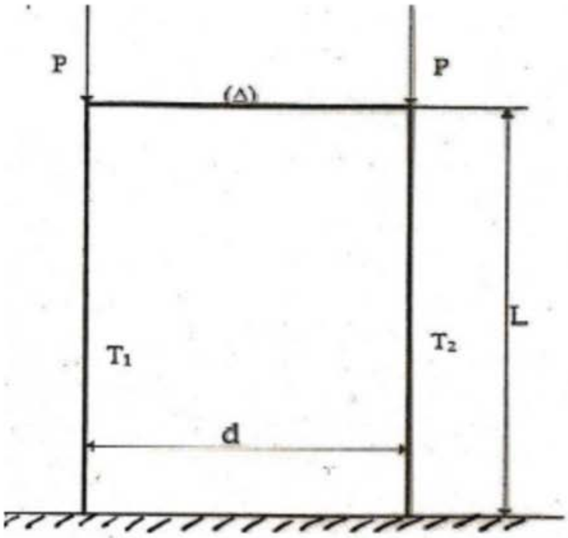


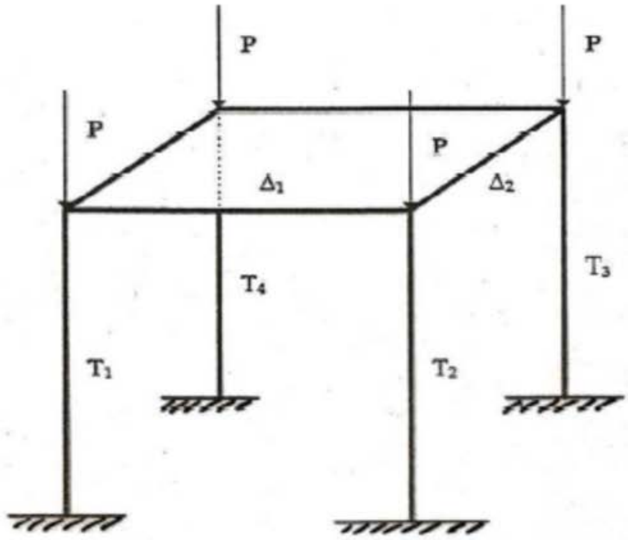
$$E(u, \theta, \varphi, \lambda)$$

$$\Longrightarrow \begin{cases} E_{,u} = 0 \\ E_{,\theta} = 0 \\ E_{,\varphi} = 0 \end{cases}$$

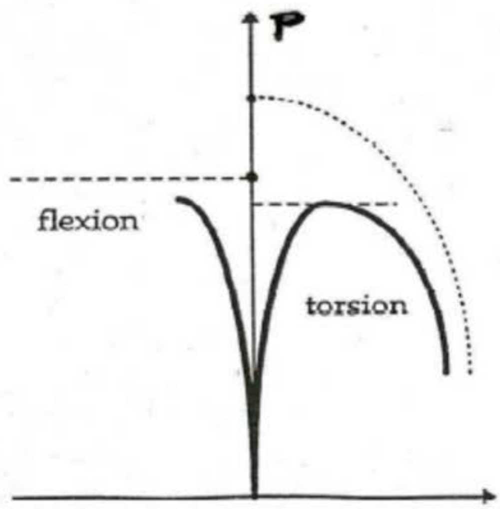


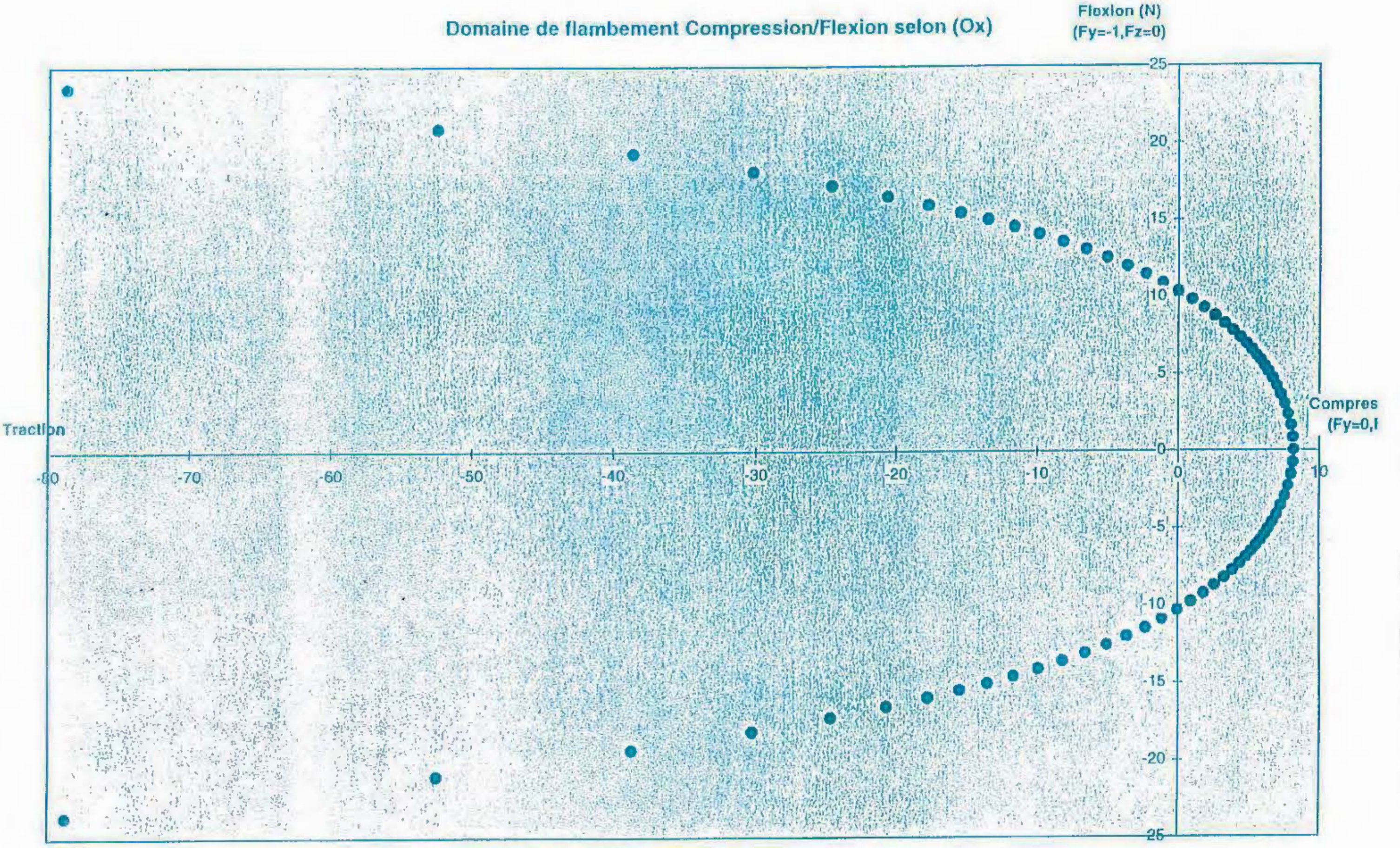
A slightly more complicated model

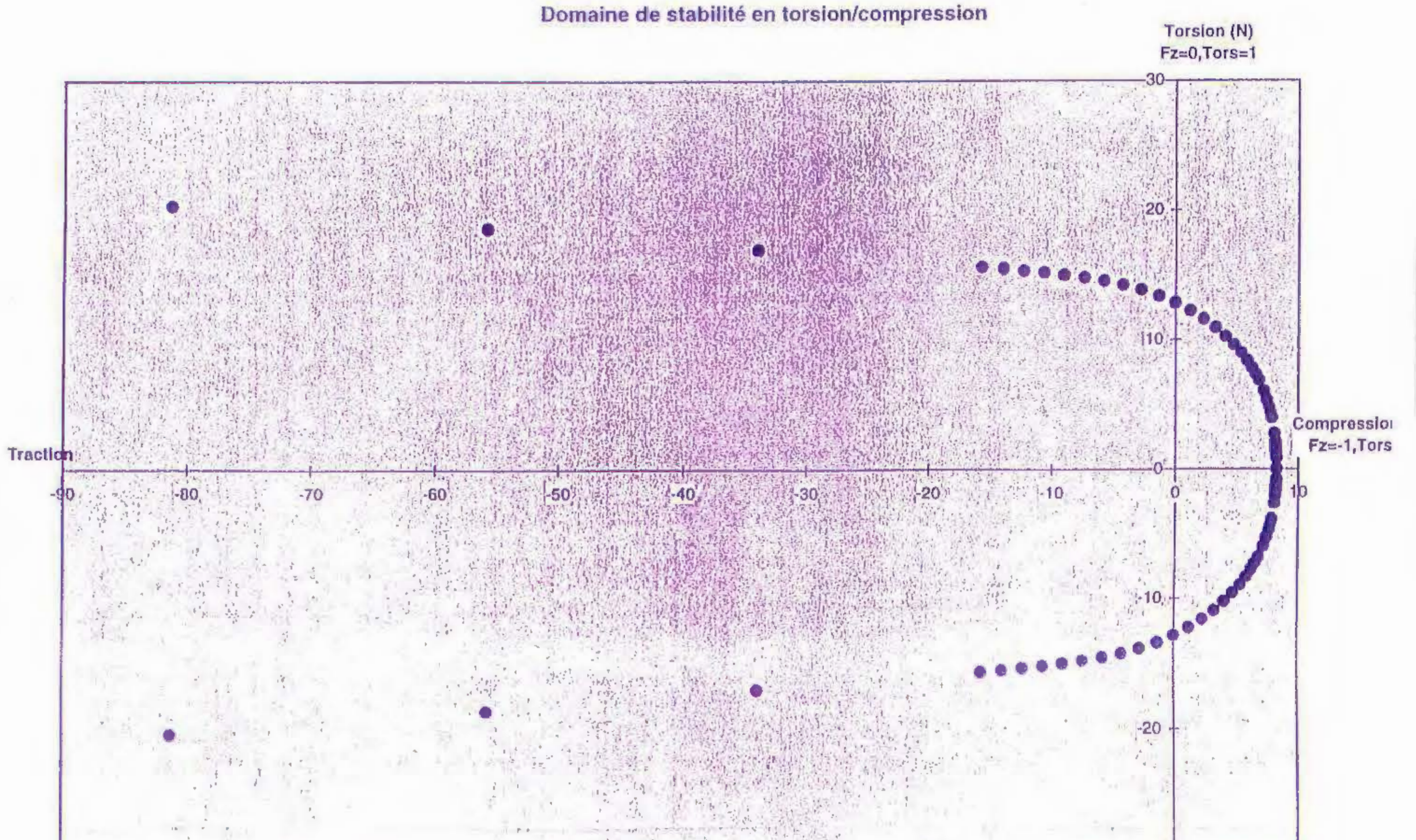




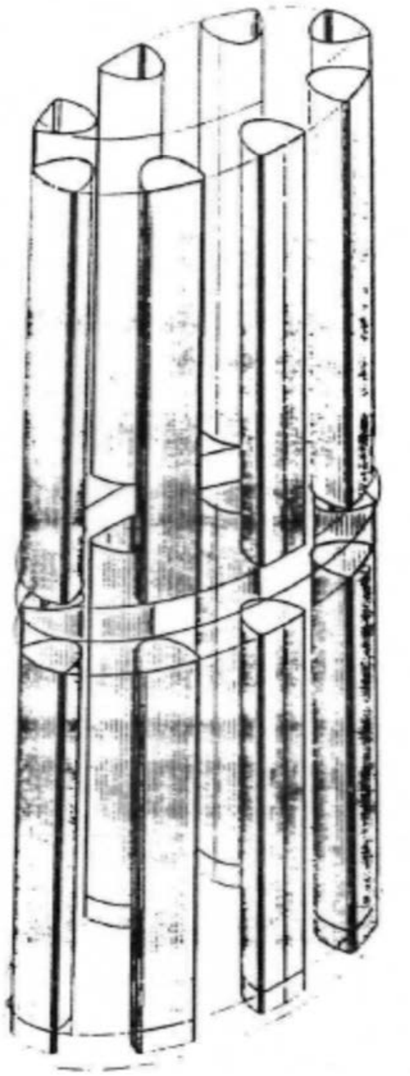
$$
\begin{cases}
P_{cr}^{Ox} \in \left]\dfrac{\pi^2}{4L^2}EI_x, \dfrac{\pi^2}{L^2}EI_x\right[\\
P_{cr}^{Oy} \in \left]\dfrac{\pi^2}{4L}EI_y, \dfrac{\pi^2}{L}EI_y\right[\\
P_{cr}^{tors} \in \left]\min\left(\dfrac{\pi^2}{L^2}EI_x, \dfrac{\pi^2}{L^2}EI_y\right), \max\left(\dfrac{\pi^2}{L^2}EI_x, \dfrac{\pi^2}{L^2}EI_y\right)\right[
\end{cases}
$$

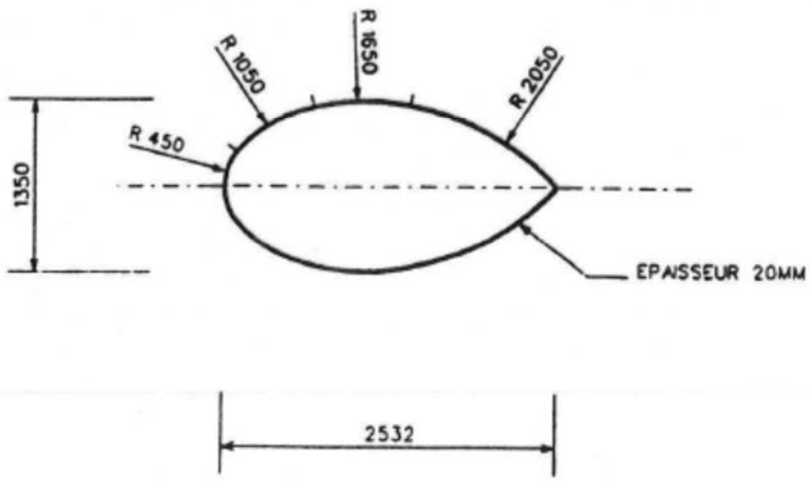



Domaine de flambement Compression/Flexion selon (Ox)
Flexion (N)
(Fy=-1,Fz=0)
Traction
Compres
(Fy=0,
25
20
15
10
5
0
-5
-10
-15
-20
-25
-70
-60
-50
-40
-30
-20
-10
0
10

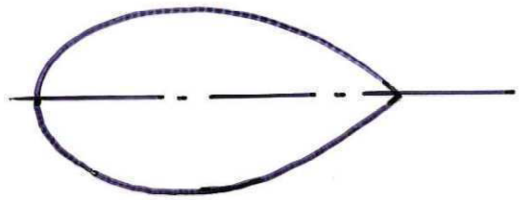

Domaine de stabilité en torsion/compression
Torsion (N)
Fz=0,Tors=1
Traction
Compressio
Fz=-1,Tors
30
20
10
0
-10
-20
-30
-80
-70
-60
-50
-40
-30
-20
-10
0
10

Choice of the modelling





Poutres Voiles Minces



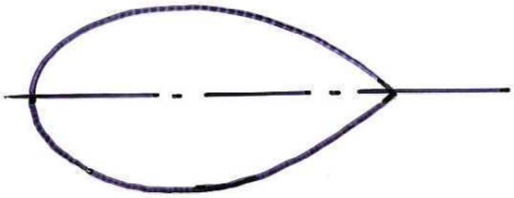
$$
\left\{
\begin{aligned}
& E I_y \, w_c^{(4)} + P \, w_c'' = 0 \\
& E I_z \, v_c^{(4)} + P \, v_c'' + P z_c \theta_x'' = 0 \\
& - M_x' + P\left(z_c^2 + \frac{I_y + I_z}{S}\right)\theta_x'' + P z_c v_c'' = 0
\end{aligned}
\right.
$$

avec $\quad M_x = G I_c \theta_x' + G(J - I_c)\chi'$

$$= G J \theta_x' - E I_\omega \chi'''$$

$+$ C.L. $\left\{ \begin{array}{l} \text{en déplacement de l'axe} \\ \text{en torsion} \\ \text{en gauchissement.} \end{array} \right.$

Poutres Voiles Minces



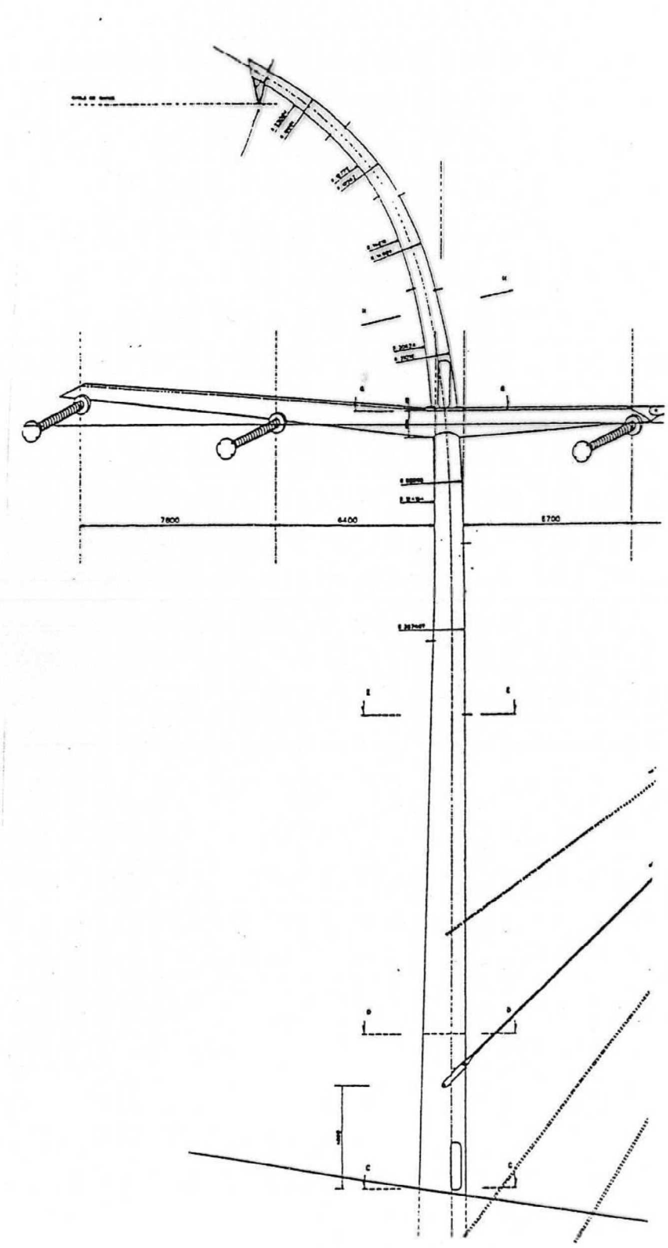
$$
\left\{
\begin{aligned}
& E I_y\, w_c^{(4)} + P\, w_c'' = 0 \\
& E I_z\, v_c^{(4)} + P\, v_c'' + P z_c\, \theta_x'' = 0 \\
& - M_x' + P\left(z_c^2 + \frac{I_y + I_z}{S}\right)\theta_x'' + P z_c\, v_c'' = 0
\end{aligned}
\right.
$$

avec $M_x = G I_c\, \theta_x' + G\,(J - I_c)\,\chi'$

$= G J\, \theta_x' - E I_\psi\, \chi'''$

+ C.L. $\left\{ \begin{array}{l} \text{en déplacement de l'axe} \\ \text{en torsion} \\ \text{en gauchissement.} \end{array} \right.$

… Ou Coque Pliée



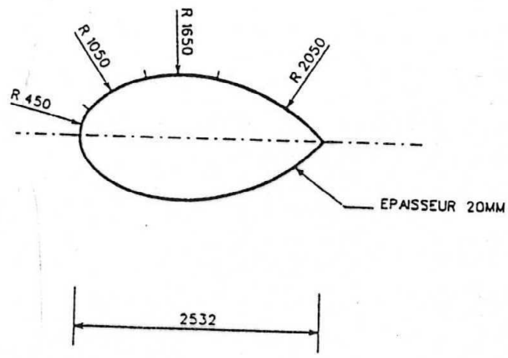
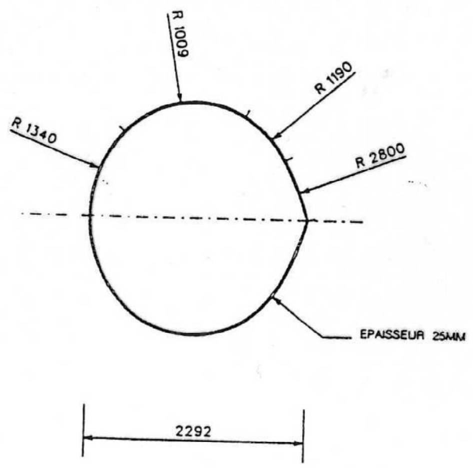
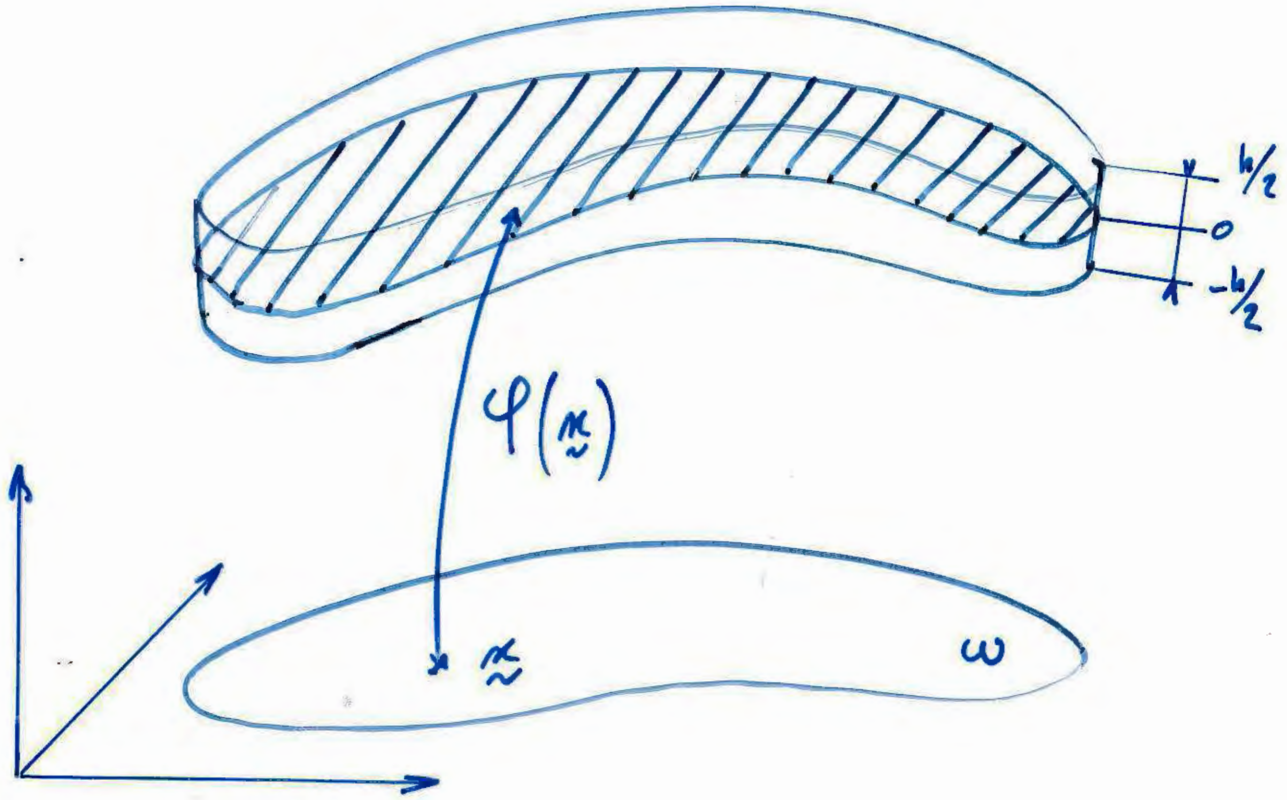


Figure 3-a: *Une section du fût en situation courante.*





Classically :

$$\begin{cases} \varphi \in \mathcal{C}^3(\omega) \\ h/L \ll 1 \;, \quad h/R_i(x) \ll 1 \end{cases}$$

Then :

$$\mathcal{A}(u,v) = \ell(v)$$

$$\swarrow$$

$$\mathcal{A}(u,v) = a^m(u,v) + h^2 a^f(u,v)$$

$$A^{3D}(u,v) = \ell(v)$$

where :

$$A^{3D}(u,v) = \int\!\!\int_{\omega}\int_{-h/2}^{h/2} G^{\alpha\beta\rho\sigma}\,\xi_{\alpha\beta}(u)\,\xi_{\rho\sigma}(v)\left[1-2Hz+Kz^2\right]\sqrt{a}\,dx\,dz$$

exactly !

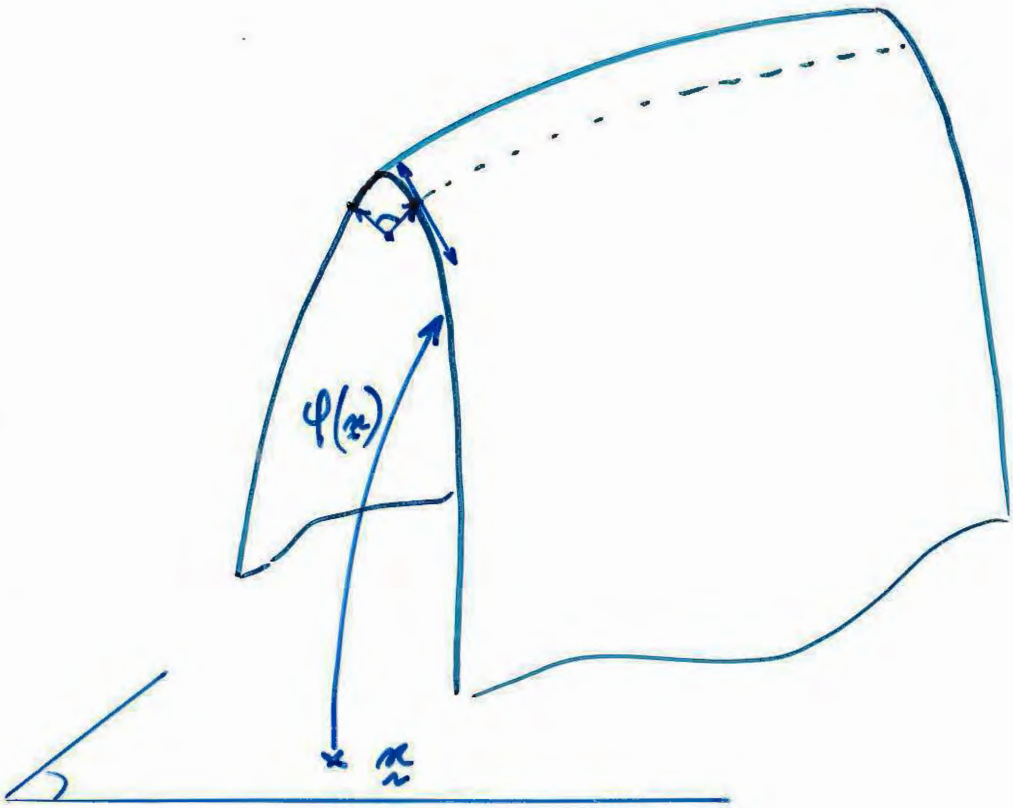
+ Integration over the cross-section

$$\Rightarrow \quad A^{2D}(u,v) = \ell(v)$$

$$\boxed{A^{2D}(u,v) = A^{m}(u,v) + A^{mf}(u,v) + A^{f}(u,v)}$$

* Coupling bending / membrane
* Same limit as "regular" models as $h \longrightarrow 0$
* Limit to a folded shell.

1^{rst} Step :



$$\varphi \in W^{2,\infty}(\omega)$$

2^{nd} step :

Assumption of very strong curvature

$$\exists\, \delta > 0, \ \min_{x \in \overline{\omega}} R_i(x) \geq \frac{h}{2} + \delta$$

(Note the difference with classical thin shells assumption).