CALCULATING BIFURCATED BRANCHES

PRELIMINARIES

Basic definitions

Fundamental tools

- Spectral Analysis
- Implicit Function Theorem

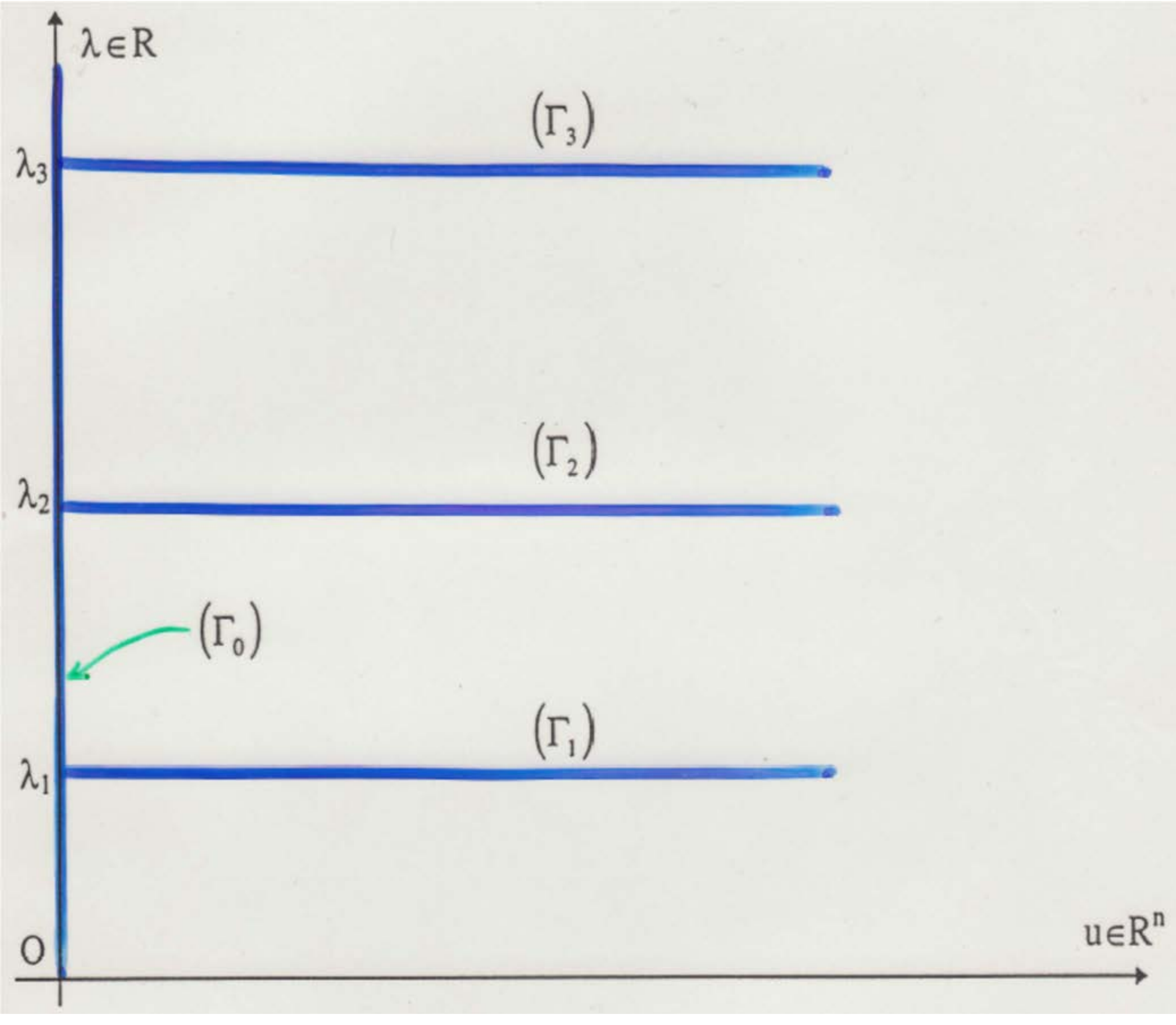
LYAPUNOV - SCHMIDT METHOD

Applications and Global Analysis

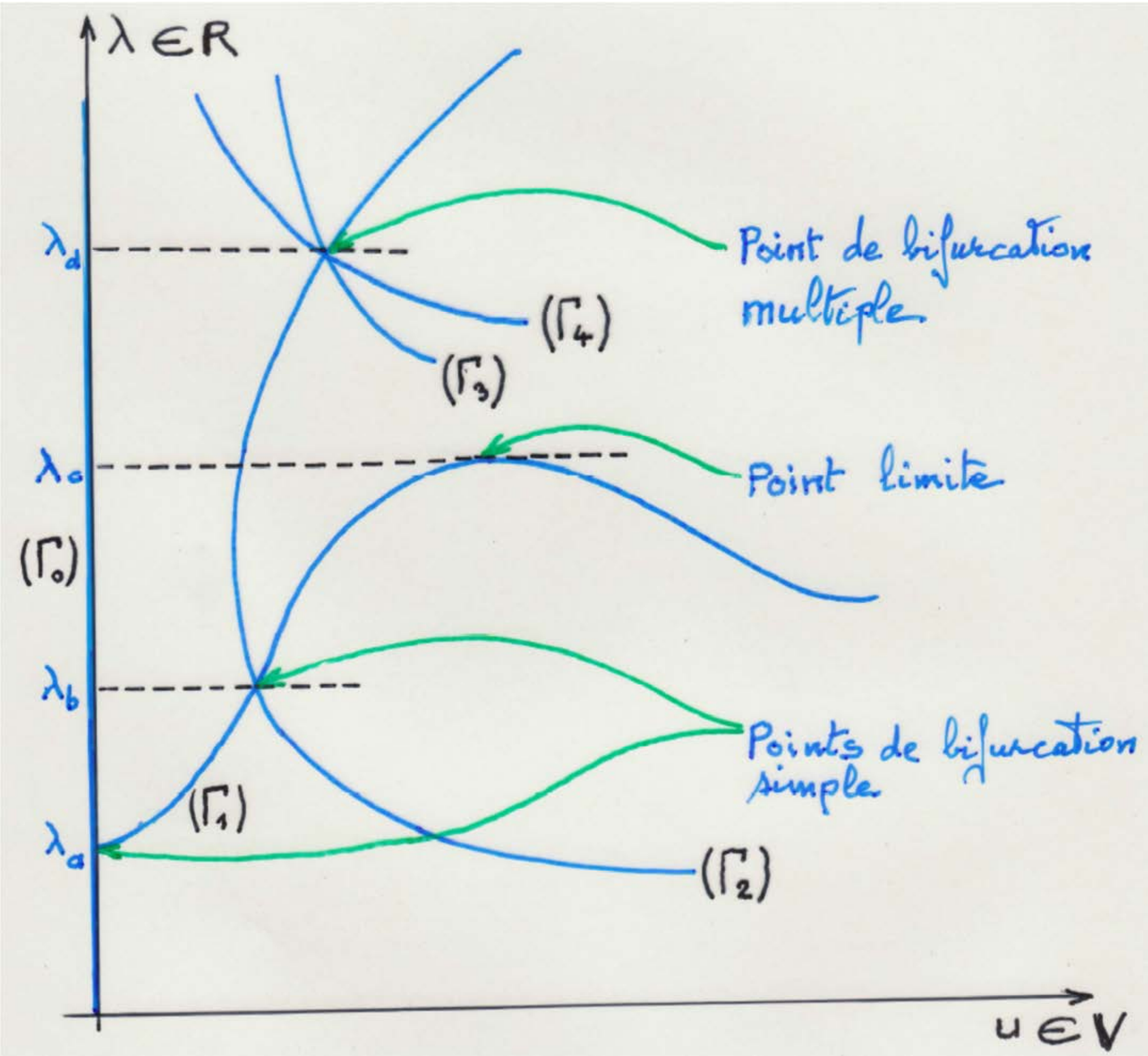
ON NUMERICAL COMPUTATIONS

SOME BASIC DEFINITIONS

$$Au - \lambda u = 0$$

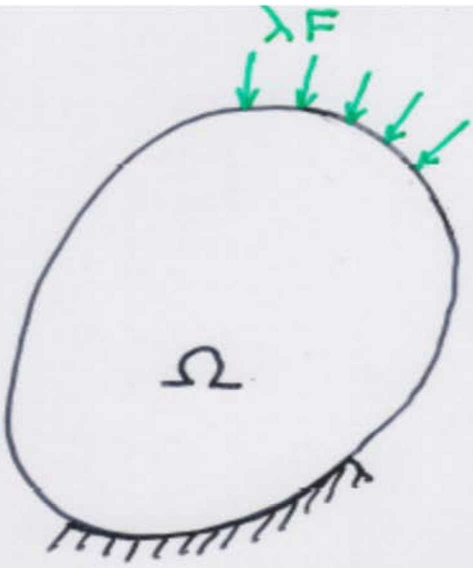


$$F(u,\lambda) = 0$$



λ ∈ R
λd
λc
(Γ0)
λb
λa
(Γ1)
(Γ2)
(Γ3)
(Γ4)
Point de bifurcation multiple
Point limite
Points de bifurcation simple
u ∈ V

FUNDAMENTAL TOOLS

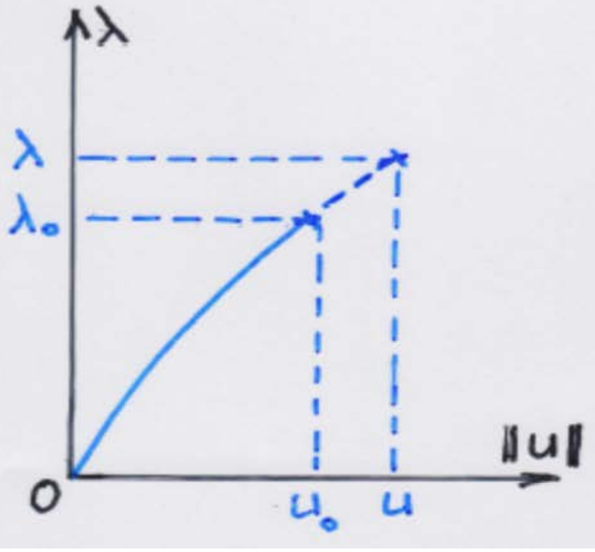


Let a conservative system, with a potential energy $E(u, \lambda)$

* An equilibrium point is a critical point of E, i.e.

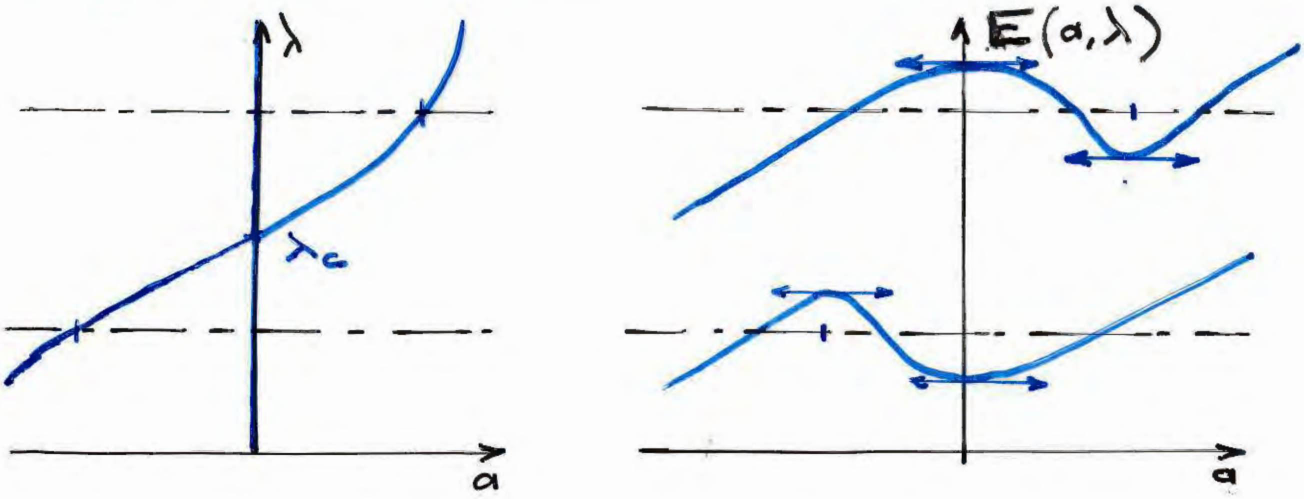
$$\langle E_{,u}(u,\lambda), v \rangle = 0, \quad \forall v \in V$$

$$V \equiv R^n \text{ or } H.$$



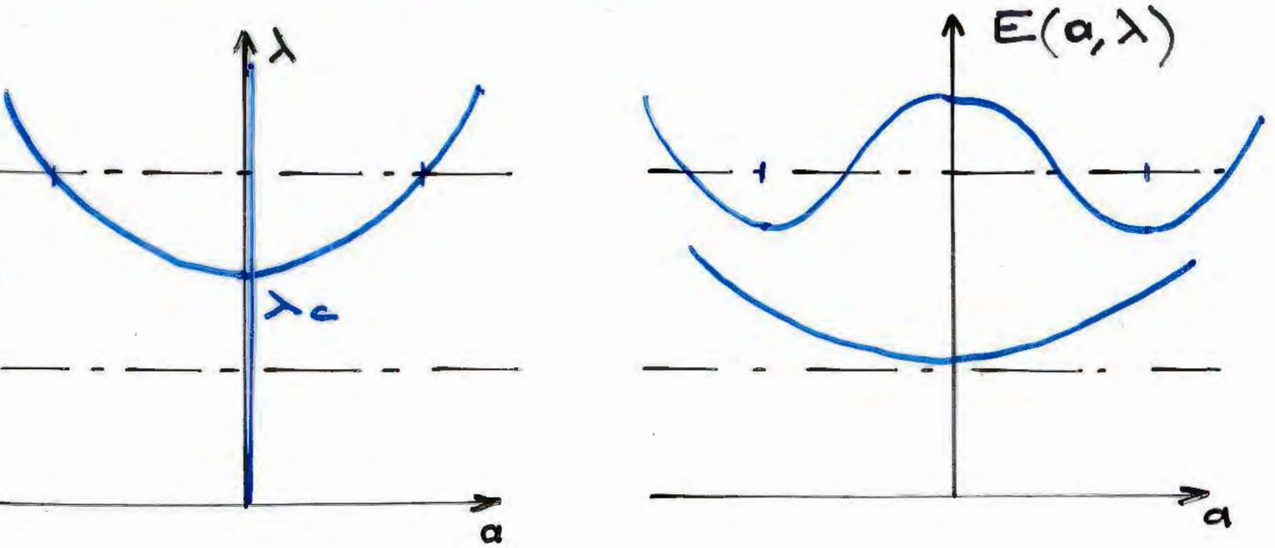
Given a solution (u_0, λ_0), can we find a solution (u, λ) for (u, λ) sufficiently close to (u_0, λ_0)?

$$\lambda - \lambda_c = C_1 a + \text{——}$$

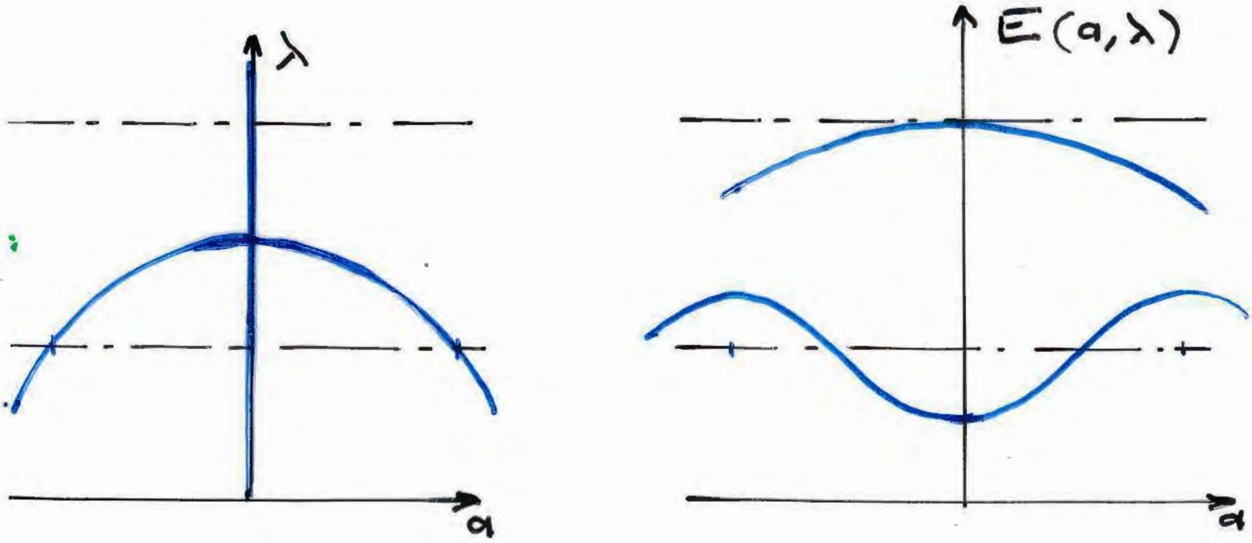


$$\lambda - \lambda_c = C_2 a^2 + \text{——}$$

$C_2 > 0$:



$C_2 < 0$:



IMPLICIT FUNCTION THEOREM

* E, F, G Banach spaces
* $F(x, y)$ continuously differentiable from $U \subset E \times F \longrightarrow G$
* $x_0, y_0 \in U$, $F(x_0, y_0) = 0$
* $F_{,y}(x_0, y_0)$ has a bounded inverse
* Then :

  $\exists!$ $y = f(x)$, continuously differentiable from a neighbourhood U_1 of x_0 to F such that

  $$y_0 = f(x_0)$$

  $$F(x, f(x)) = 0 \quad \forall x \in U_1 .$$

Corollary

F analytical from $U \subset E \times F$ to G

then :

$y = f(x)$ analytical in a neighbourhood of x_0.

$$\text{Let} \quad \mathcal{F}(u,\lambda) \equiv E_{,u}(u,\lambda)$$

DEFINITIONS

Let $\mathcal{S}$ be the set of solutions of $\mathcal{F}(u,\lambda)=0$, i.e.

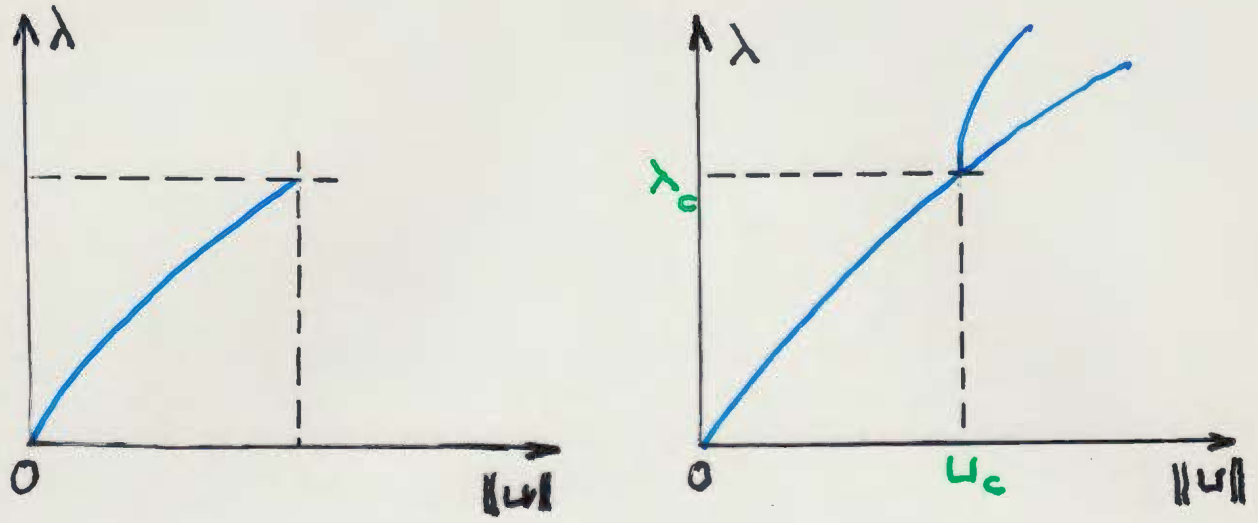
$$\mathcal{S} = \{(u,\lambda) \in V \times R \,/\, \mathcal{F}(u,\lambda)=0\}$$

1°) A solution branch is a parametrized curve $\mathcal{B} = \{(u(s), \lambda(s)),\ s \in \Delta \subset R\}$ such that $\mathcal{B} \subset \mathcal{S}$, $s \longrightarrow (u(s), \lambda(s))$ is continuous.

2°) $(u_0, \lambda_0) \in \mathcal{B}$ is a bifurcation point if any neighbourhood of (u_0, λ_0) contains one solution which does not belong to $\mathcal{B}$.

3°) Singular point (u_c, λ_c)

$$\exists\, U \in V,\ \langle E_{,uu}(u_c,\lambda_c)U, v\rangle = 0 \quad \forall v \in V.$$



PROPOSITIONS

Let an equilibrium problem $F(u,\lambda)=0$, $F: V\times R \longrightarrow V$, such that $(0,\lambda)$ is a solution for any λ.

1°) If $F_{,u}(0,\lambda)$ is an isomorphism of V, then $(0,\lambda)$ is not a bifurcation point.

2°) If $(0,\lambda_0)$ is a bifurcation point of F, then λ_0 belongs to the spectrum of $F_{,u}(0,\lambda_0)$, i.e. $F_{,u}(0,\lambda_0)$ does not have a bounded inverse.

SPECTRAL ANALYSIS

A : Linear bounded operator $H \longrightarrow H$

λ regular value if $(A - \lambda I)^{-1}$ exists, bounded and defined everywhere on H

Resolvant set $\rho(A)$:

$$\{\lambda / \lambda \text{ regular values of } A\}$$

Spectrum of A:

$$\sigma(A) = \text{complementary of } \rho \text{ in } \mathbb{C}$$

Point spectrum :

$$\sigma_p(A) = \{\lambda / \exists u \in H,\ u \neq 0,\ (A - \lambda I) u = 0\} = \{\text{Eigenvalues of } A\}$$

Continuous spectrum :

$$\sigma_c(A) = \{\lambda / \cdot (A - \lambda I) u = 0 \Rightarrow u = 0,\ \cdot (A - \lambda I)^{-1} \text{ unbounded on } R_\lambda(A),\ \cdot R_\lambda(A) \text{ dense in } H\}$$

Residual spectrum :

$$\sigma_r(A) = \Big\{ \lambda \,/\, \cdot (A - \lambda 1) u = 0 \Rightarrow u = 0,$$

$$\cdot (A - \lambda 1)^{-1} \text{ unbounded on } R_\lambda(A),$$

$$\cdot \; R_\lambda(A) \text{ not dense in } H \Big\}$$

THEOREMS :

* $\rho(A)$ is open.
* $\sigma(A) = \sigma_p \cup \sigma_c \cup \sigma_r \,, \neq \phi$.
* A self adjoint : $\sigma(A)$ real, $\sigma_r(A) = \phi$.
* A compact : $\lambda \neq 0, \lambda \in \sigma(A) \Rightarrow \lambda \in \sigma_p(A)$.

DEFINITION :

$$\sigma(A) = \sigma_d \cup \sigma_{ess}$$

$$\sigma_{ess}(A) = \Big\{ \lambda \,/\, \lambda \text{ is not an isolated eigenvalue of finite multiplicity} \Big\}$$

POST-BUCKLING ANALYSIS

Lyapunov Schmidt method

* Equilibrium equation $\langle E_{,u}(0,\lambda), v \rangle = 0$

* Assume $\exists U$ for λ_c such that:

$$\langle E_{,uu}(0,\lambda_c)\, U, v \rangle = 0, \ \forall v \in H$$

* $$\begin{cases} \lambda = \lambda_c + \mu \\ u = \xi U + w \end{cases} \quad \text{with} \begin{cases} U \in \mathcal{N} \\ w \in \mathcal{N}^\perp \end{cases}$$

* Then the equilibrium equation gives

$$(1) \quad \langle E_{,u}(\xi U + w, \lambda_c + \mu), v \rangle = 0, \ \forall v \in \mathcal{N}$$

$$(2) \quad \langle E_{,u}(\xi U + w, \lambda_c + \mu), \tilde{v} \rangle = 0, \ \forall \tilde{v} \in \mathcal{N}^\perp$$

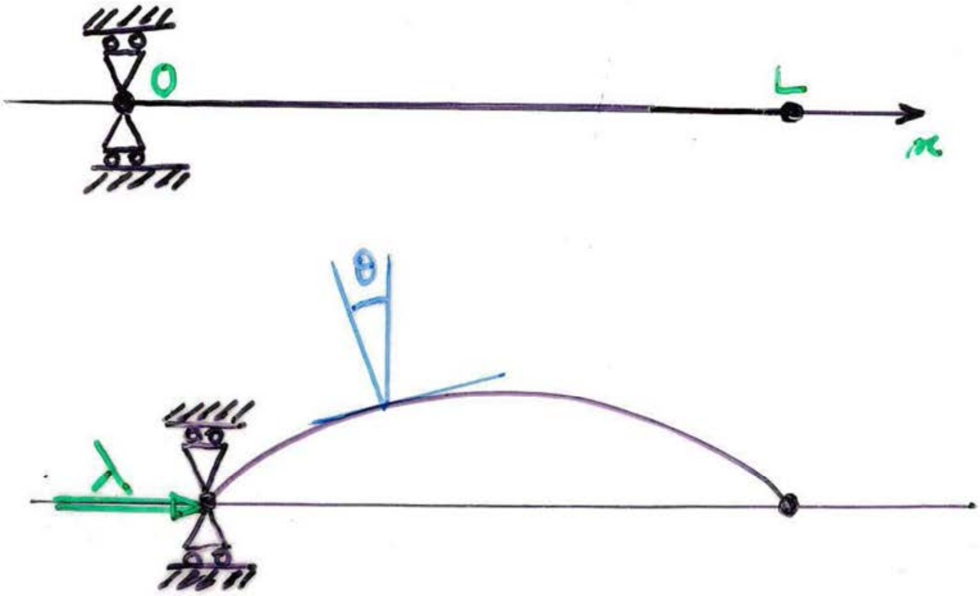
* $$(2) \implies \exists!\, w = w(\xi U, \mu)$$

* Bifurcation equation:

$$\langle E_{,u}(\xi U + w(\xi U, \mu), \lambda_c + \mu), v \rangle = 0$$

$\forall v \in \mathcal{N}$.

COMPRESSED BEAM : the elastica model



Déplacement repéré par $\theta = \theta(x)$

* $$E(\theta, \lambda) = \int_0^L \frac{EI}{2} \theta'^2 \, dx - \lambda \left[L - \int_0^L \cos\theta \, dx \right]$$

* $$\langle E_{,\theta}(\theta, \lambda), \varphi \rangle = \int_0^L EI \theta' \varphi' \, dx - \lambda \int_0^L \sin\theta . \varphi \, dx$$

$$\hookrightarrow \left| \begin{array}{l} EI \theta'' + \lambda \sin\theta = 0 \\ \theta'(0) = \theta'(L) = 0 \end{array} \right.$$

$$* \left\langle E_{,\theta\theta}(\theta,\lambda)\cdot\varphi,\psi\right\rangle = \int_0^L EI\,\psi''\varphi + \lambda\int_0^L \psi\cos\theta\cdot\varphi\,dx$$

$$\left|\begin{array}{l} EI\,U'' + \lambda_c U = 0 \\ U'(0) = 0\ ,\ U'(L) = 0 \end{array}\right.$$

$$U(x) = \cos\frac{\pi x}{L}$$

$$\lambda_c = \frac{\pi^2 EI}{L^2}$$

Calcul post-critique :

$$F(\theta,\lambda) \equiv L(\theta,\lambda) + r(\theta,\lambda) = 0$$

avec :
$$\left\{\begin{array}{l} L(\theta,\lambda) = L(\lambda)\cdot\theta \\ \qquad\qquad = EI\,\theta'' + \lambda\theta \\ r(\theta,\lambda) = \lambda(\sin\theta - \theta) \\ \qquad\qquad = o(\|\theta\|^3) \end{array}\right.$$

$$* \quad \lambda < \lambda_c : \quad \theta = -L(\lambda)^{-1} r(\theta,\lambda)$$

* Etude locale autour de $\lambda = \lambda_c$.

- Décomposition de l'inconnue
- Projection de l'équation.

$$\boxed{\begin{aligned} &\theta(x) = a\,U + v(x) \\ &a \in \mathbb{R}\ ,\ v \in U^\perp \end{aligned}}$$

Projection sur $U^\perp$:

$$P\,\theta(x) = \theta(x) - \frac{U(x)\int_0^L \theta(x)\,U(x)\,dx}{\int_0^L U(x)^2\,dx}$$

1°/ $$g(a, v, \lambda) = L(\lambda_c)v + P\,r(aU+v, \lambda) + P\big(L(\lambda) - L(\lambda_c)\big)(aU+v)$$

2°/ $$h(a, v, \lambda) = \int_0^L \Big\{L(\lambda)(aU+v) + r(aU+v, \lambda)\Big\}U\,dx$$

1°/ $g : R \times U^{\perp} \times R \longrightarrow U^{\perp}$

est inversible localement,

car $g_{,v}(0, 0, \lambda_c) = L(\lambda_c)$

$$\boxed{v = \hat{v}(a, \lambda) = \mathcal{O}(a^3) + \mathcal{O}((\lambda - \lambda_c)a)}$$

Solution unique.

2°/

$$h = h(a, \lambda) = \int_0^L \left\{ L(\lambda)(aU + \hat{v}) + r(aU + \hat{v}, \lambda) \right\} U \, dx$$

avec $L(\lambda) U = (\lambda - \lambda_c) U$

$$\begin{cases} h(a, \lambda) = (\lambda - \lambda_c) a \int_0^L U^2 dx \\ \qquad - \lambda a^3 \int_0^L \frac{1}{6} U^4 dx \\ \qquad + \text{t.o.s.} \end{cases}$$

$$\begin{cases} a = 0 \quad \forall \lambda \in R \\ \frac{\lambda - \lambda_c}{\lambda_c} = \frac{a^2}{8} + \mathcal{O}(a^4) \end{cases}$$

$$* \quad E(\theta,\lambda) = E_{,\theta\theta}(\theta,\lambda_c) + (\lambda-\lambda_c)\frac{d}{d\lambda}E_{,\theta\theta}(\theta,\lambda_c) + E_{,\theta\theta\theta\theta}(\theta,\lambda_c) + \text{t.o.s.}$$

$$\text{t.o.s} = O\left[(\lambda-\lambda_c)^2\theta^2 + (\lambda-\lambda_c)\theta^4 + \theta^6\right]$$

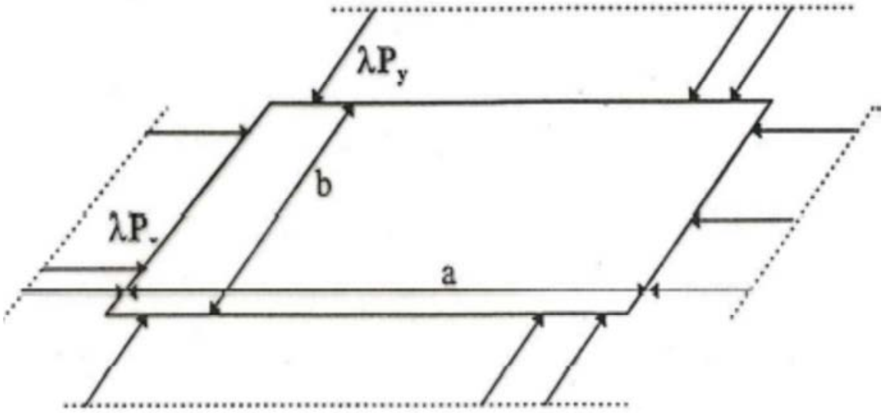
Lyapunov . Schmidt $\Rightarrow \theta(x) = a\cos\frac{\pi x}{L} + O(a^3)$

On tronque : $\theta(x) = a\cos\frac{\pi x}{L}$

$$\hookrightarrow E(aU,\lambda) = a^2(\lambda-\lambda_c)\frac{d}{d\lambda}E_{,\theta\theta}(U,\lambda_c) + a^4 E_{,\theta\theta\theta\theta}(U,\lambda_c)$$

$$\left[\begin{array}{l} \lambda - \lambda_c = C_1 a + C_2 a^2 + \text{——} \\ \\ C_1 = 0. \\ \\ C_2 = -\dfrac{2E_{,\theta\theta\theta\theta}(U,\lambda_c)}{\frac{d}{d\lambda}E_{,\theta\theta}(U,\lambda_c)} \end{array}\right.$$

RECTANGULAR PLATE



Local analysis :

$$w(x,y) = \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}$$

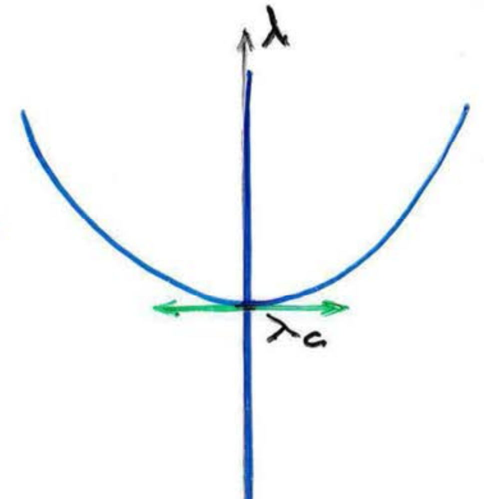
$$\lambda_c = k\pi^2 D, \quad k = \frac{\left[(m/a)^2 + (n/b)^2\right]^2}{(m/a)^2 P_x + (n/b)^2 P_y}$$

Then (Lyapunov - Schmidt)

$$\lambda - \lambda_c = C_1 w_{max} + C_2 w_{max}^2 + \cdots$$

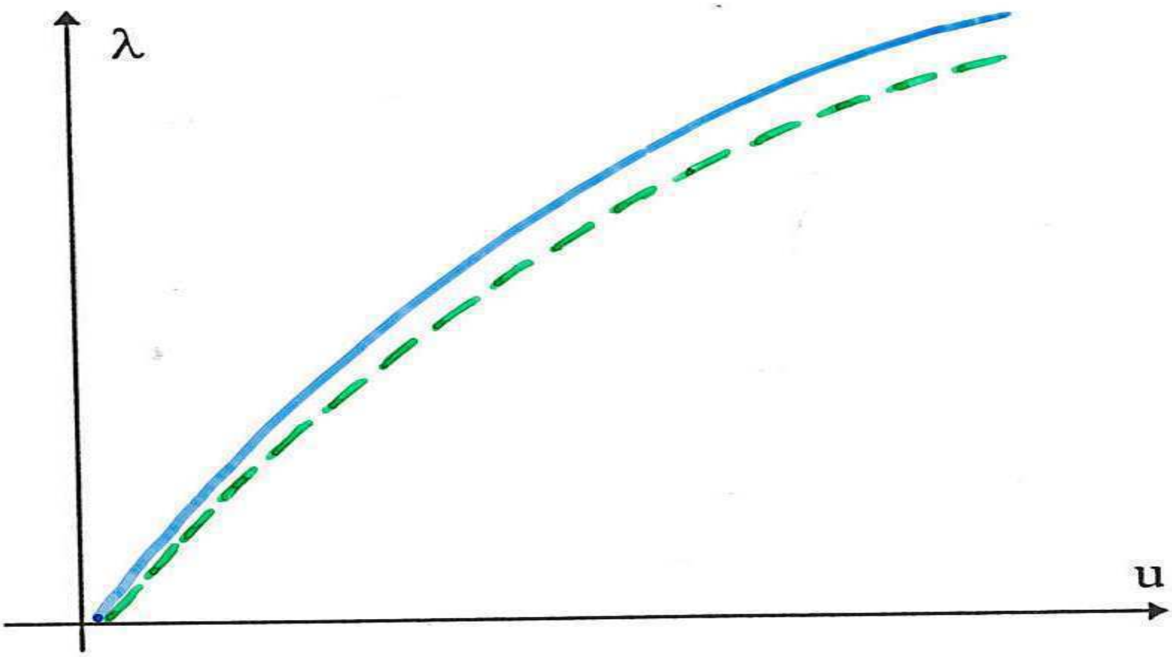
$$C_1 = 0$$

$$C_2 = \frac{3}{2}\lambda_c(1-\nu^2)\frac{\left(\frac{m}{a}\right)^4 + \left(\frac{n}{b}\right)^4}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

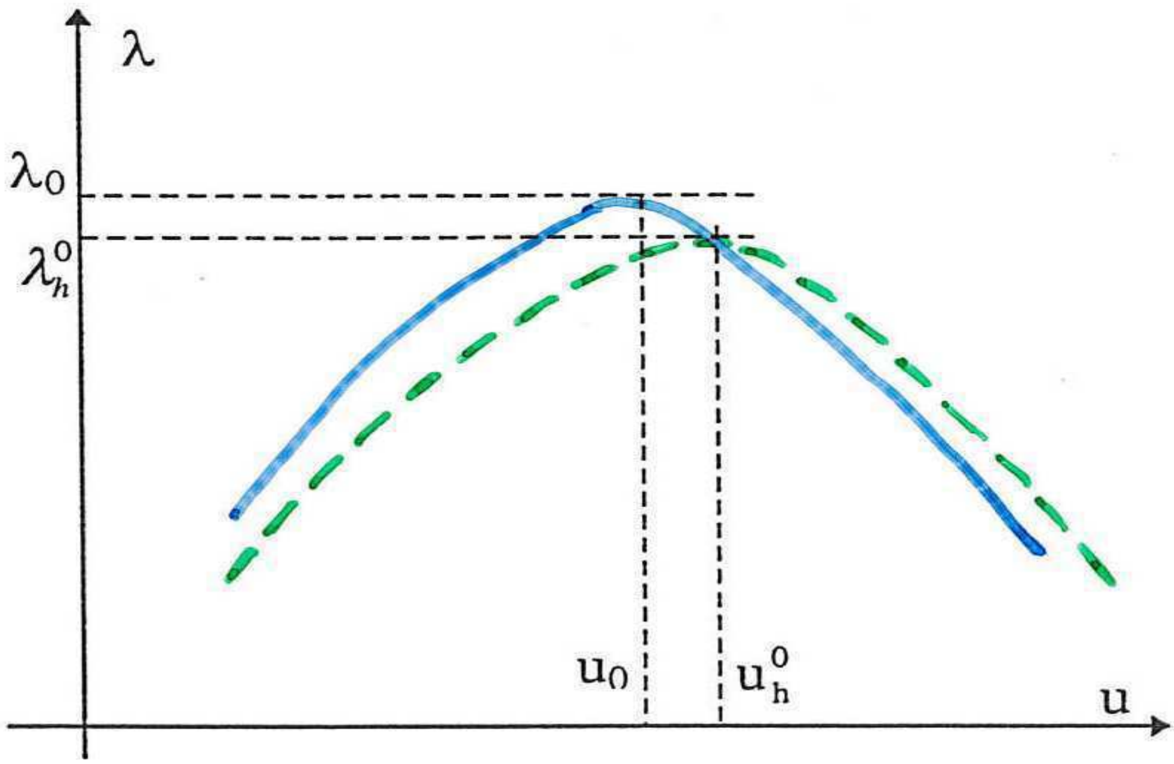


ON NUMERICAL COMPUTATIONS

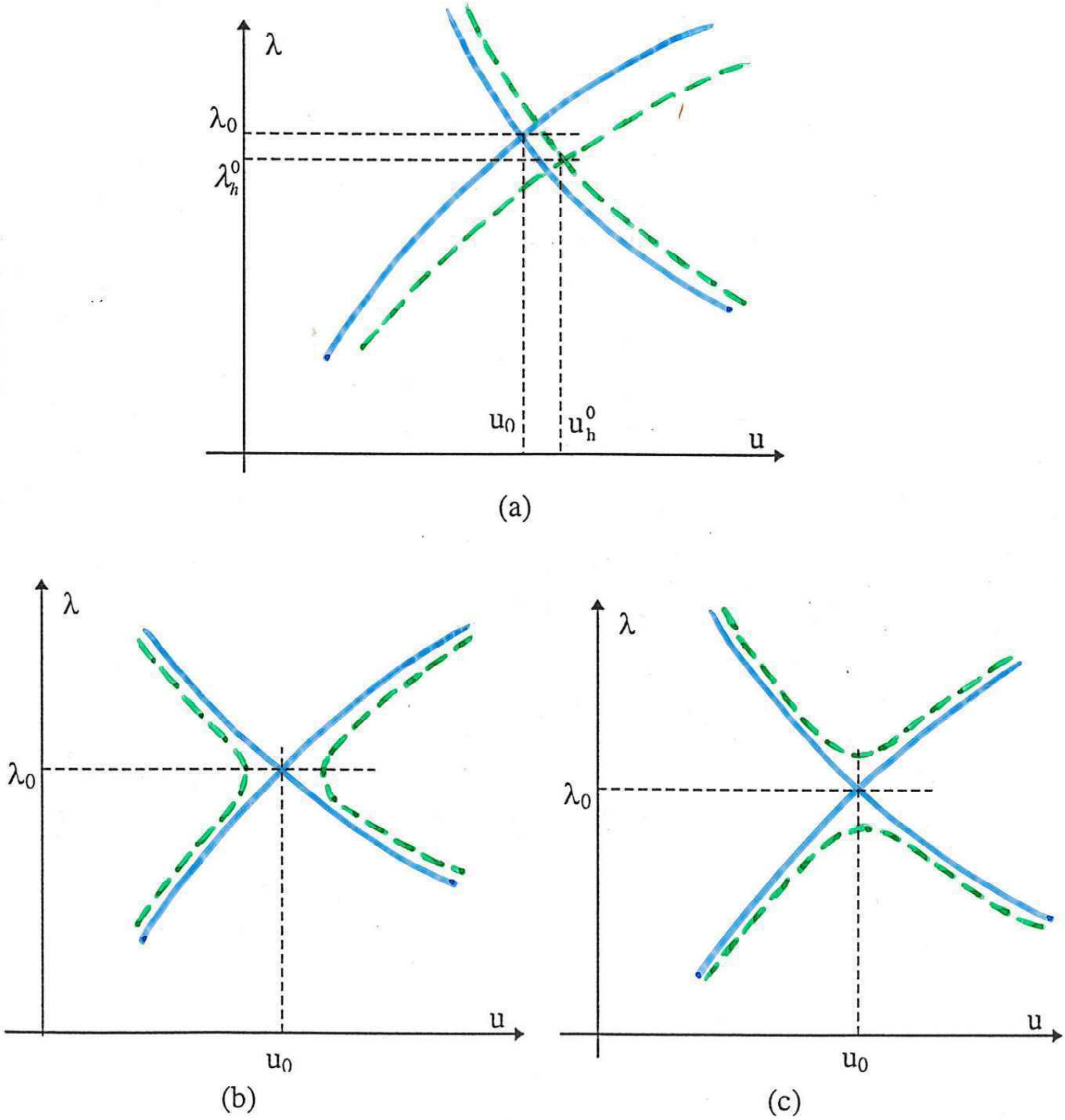
Approximation of a Regular Branch



Approximation of a Limit Point



Approximation of a Simple Bifurcation Point



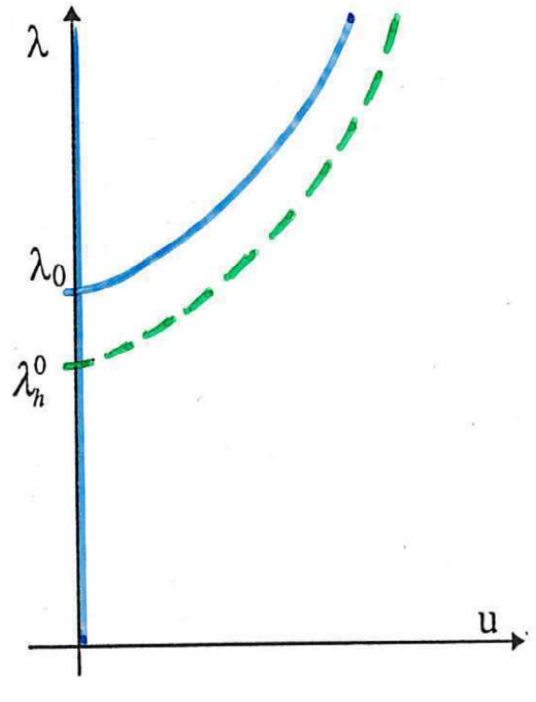
Hypothèse :

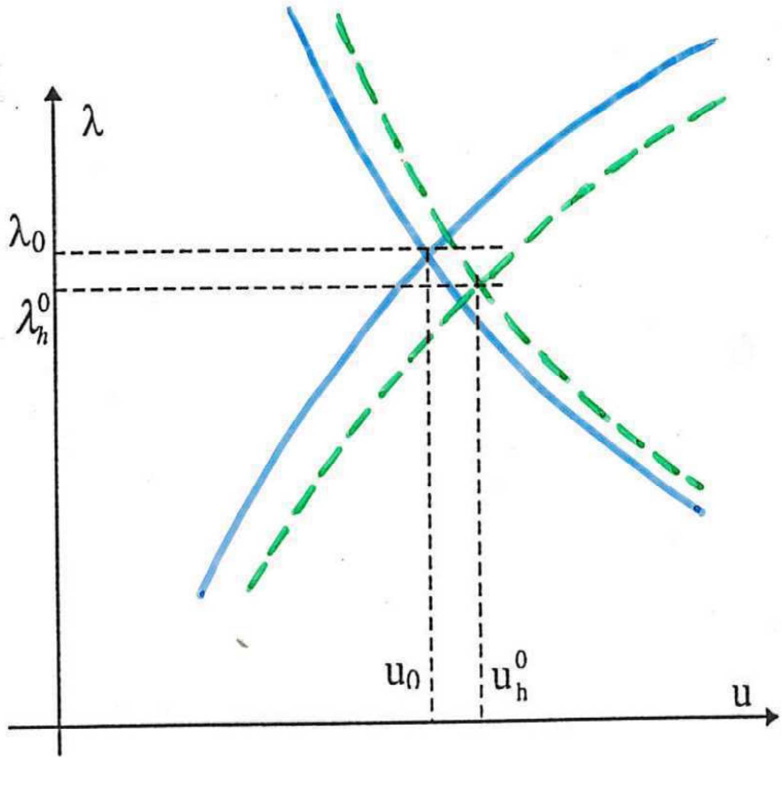
$F(u, \lambda)$ est telle que

* Soit $F(0, \lambda) = 0 \quad \forall \lambda \in \Lambda$

* Soit $\exists \mathcal{R}$ & $\begin{cases} - \mathcal{R} \neq Id \\ - \mathcal{R}^2 = Id \\ - F(\mathcal{R}u, \lambda) = F(u, \lambda) \end{cases}$

Alors :





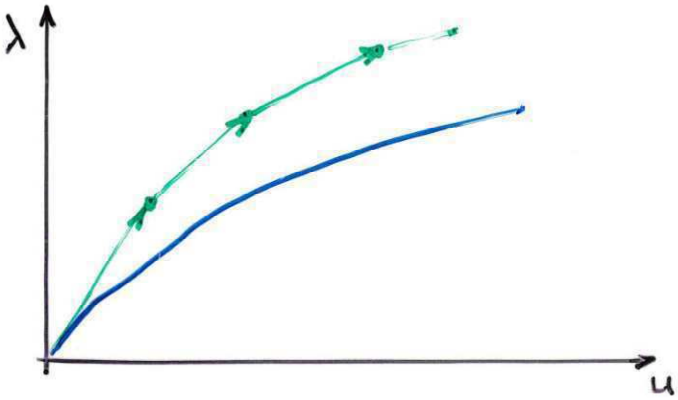
Path Following (or Continuation) Method

(*) $F(u, \lambda) = 0$

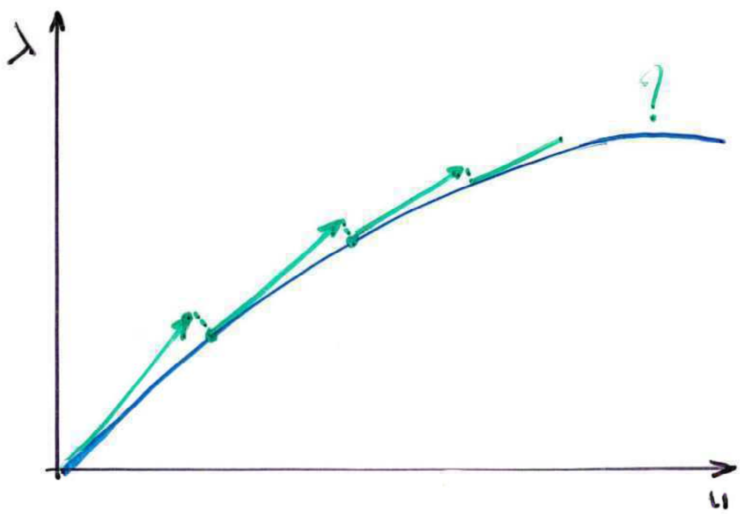
↳ Objectif $u = u(\lambda)$

1°) Prédiction : Euler

2°) Correction : Newton



Euler seulement



Euler + Newton

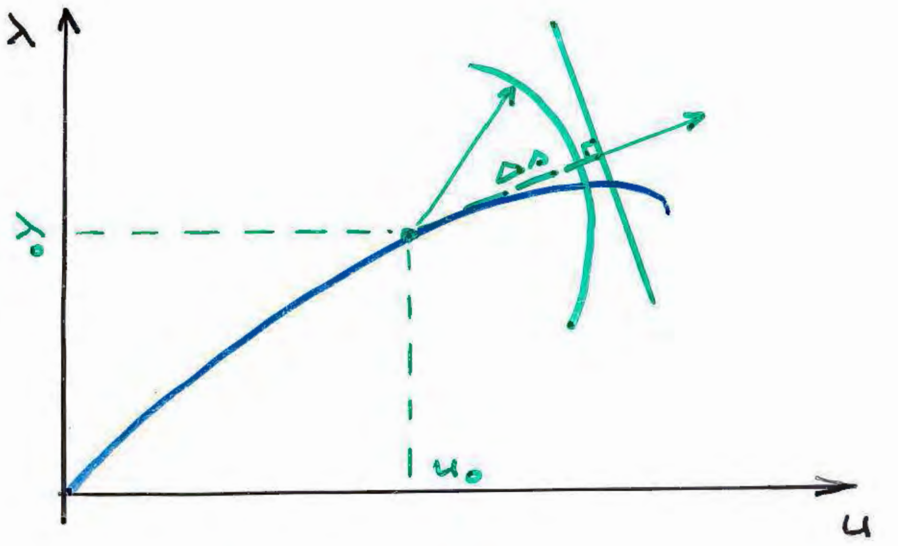
$$\cancel{u = u(\lambda)} \longrightarrow \{u = u(s),\ \lambda = \lambda(s)\}$$

(*) remplacé par (**)

$$(**)\ \begin{cases} F(u(s), \lambda(s)) = 0 \\ N(u(s), \lambda(s), s) = 0 \end{cases}$$

$$\hookrightarrow \|\dot{u}(s)\|^2 + \dot{\lambda}(s)^2 = 1 \quad (1)$$

$$(u_1 - u_0)^{*} \cdot \dot{u}_0 + (\lambda_1 - \lambda_0)\dot{\lambda}_0 - \Delta s = 0$$



Techniquement on remplace (1) par (2)

$$\theta \|\dot{u}(s)\|^2 + (1-\theta)\dot{\lambda}(s)^2 = 1$$

On peut ainsi

* calculer les branches régulières
** passer les points limites
*** "sauter" les points de bifurcation

* et ** résultent de :

$$P(x, s) \equiv \begin{pmatrix} F(u, \lambda) \\ N(u, \lambda, s) \end{pmatrix}$$

Alors :

$$P_{,x} = \begin{pmatrix} F_{,u}(u(s), \lambda(s)) & F_{,\lambda}(u(s), \lambda(s)) \\ N_{,u}(u(s), \lambda(s), s) & N_{,\lambda}(u(s), \lambda(s), s) \end{pmatrix}$$

est non singulière

De plus, connaissant $(x(s_0), s_0)$:

$$\dot{x}(s_0) = - P_{,x}(x(s_0), s_0)^{-1} \cdot P_{,s}(x(s_0), s_0)$$

1°/ Signe de $\Delta s \Longleftrightarrow$ sens de suivi de branche

2°/ Calcul d'une solution :

$\Rightarrow$ Homotopie.

3°/ Plusieurs paramètres.

$$F(u, \lambda, \mu) = 0$$

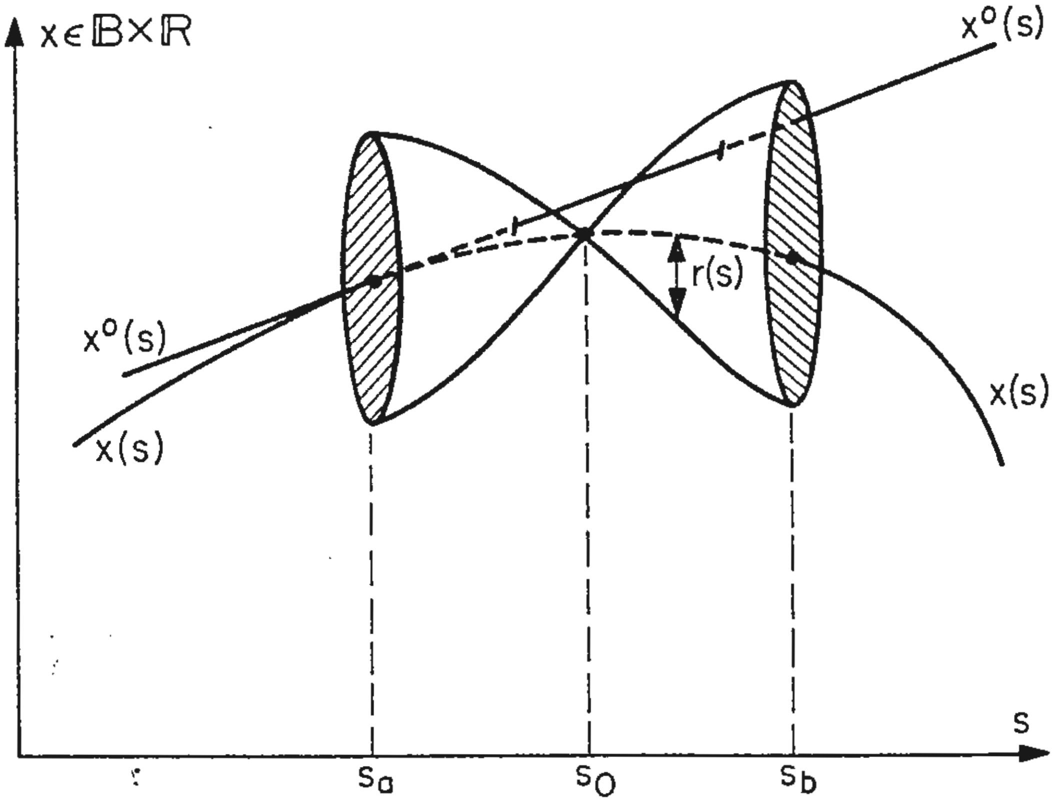
1ère étape : μ fixé $\Rightarrow$ pt. singulier.

2ième étape : continuation du pt singulier.

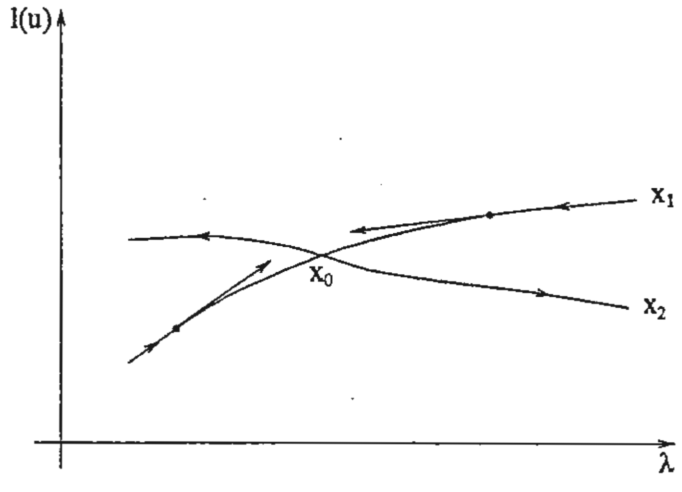
$$\begin{cases} - \begin{cases} F(u, \lambda, \mu) = 0 \\ F_{,u}(u, \lambda, \mu).v = 0 \end{cases} \\ - (u_1 - u_0)^* \dot{u}_0 + (v_1 - v_0)^* \dot{v}_0 + (\lambda_1 - \lambda_0)\dot{\lambda}_0 + (\mu_1 - \mu_0)\dot{\mu}_0 - \Delta s = 0 \end{cases}$$

4°/ Nombre de solutions?

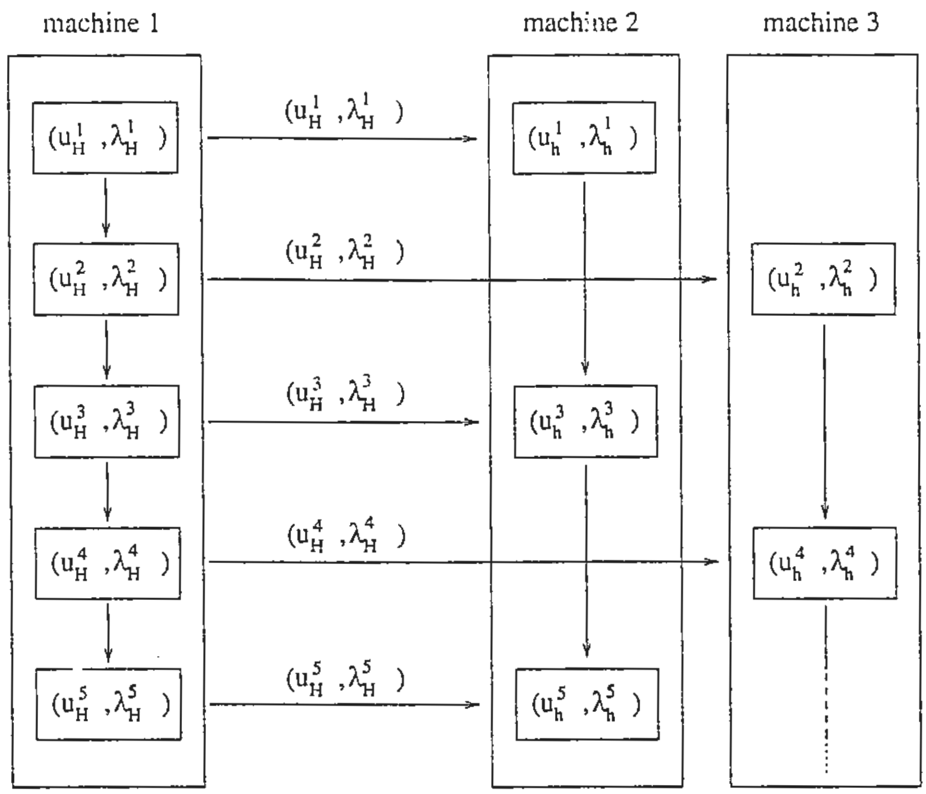
5°/ Parallélisation

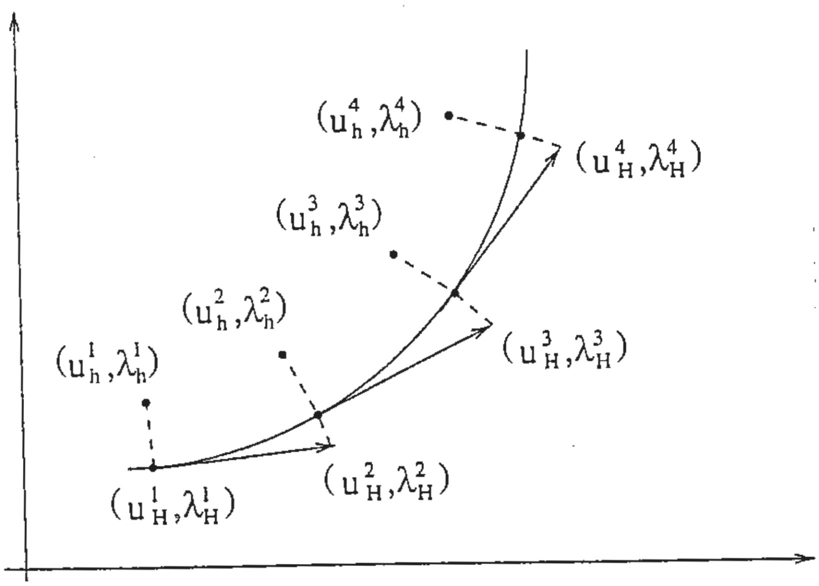


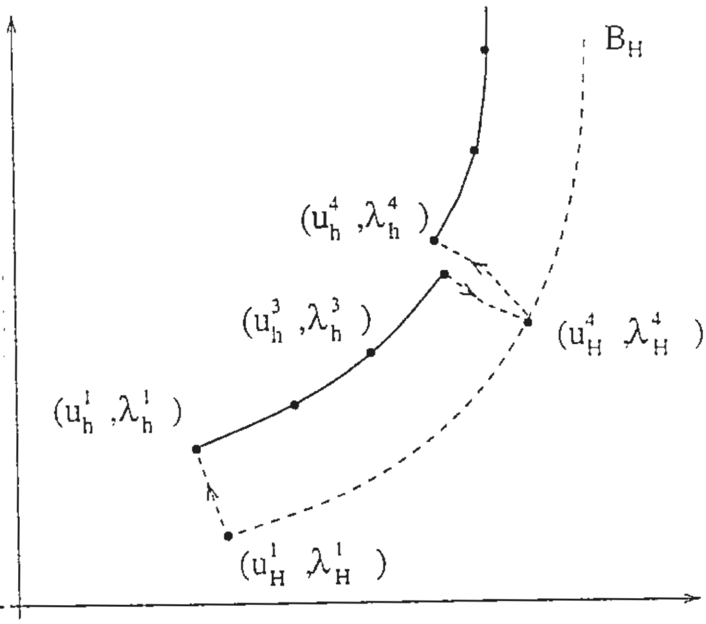
x ∈ 𝔹×ℝ
$x^o(s)$
$r(s)$
$x^o(s)$
$x(s)$
$x(s)$
s
s_a
s_0
s_b

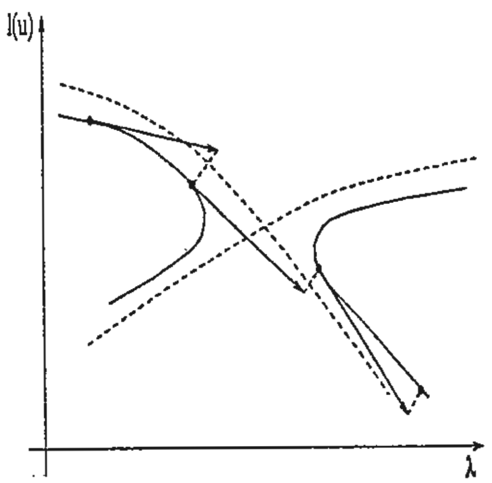


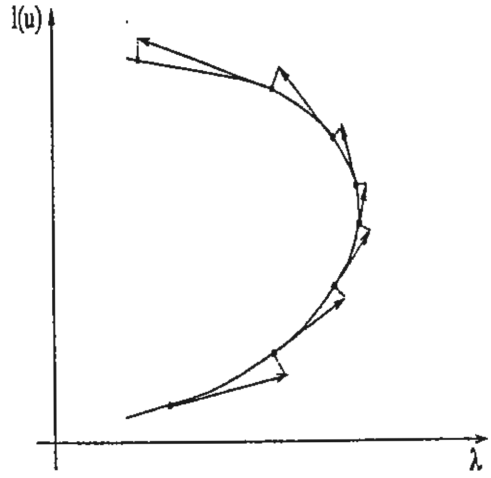
I(u)
x_1
x_0
x_2
λ

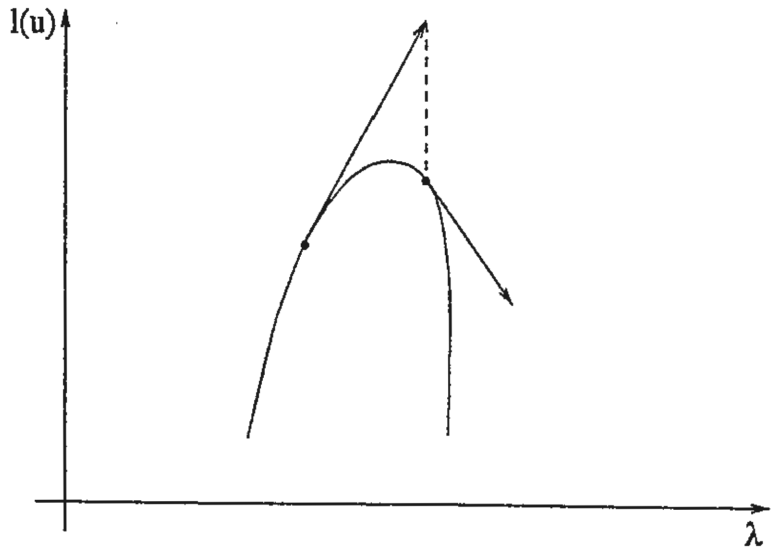

machine 1
machine 2
machine 3
(u_H^1, λ_H^1)
(u_H^2, λ_H^2)
(u_H^3, λ_H^3)
(u_H^4, λ_H^4)
(u_H^5, λ_H^5)
(u_h^1, λ_h^1)
(u_h^2, λ_h^2)
(u_h^3, λ_h^3)
(u_h^4, λ_h^4)
(u_h^5, λ_h^5)

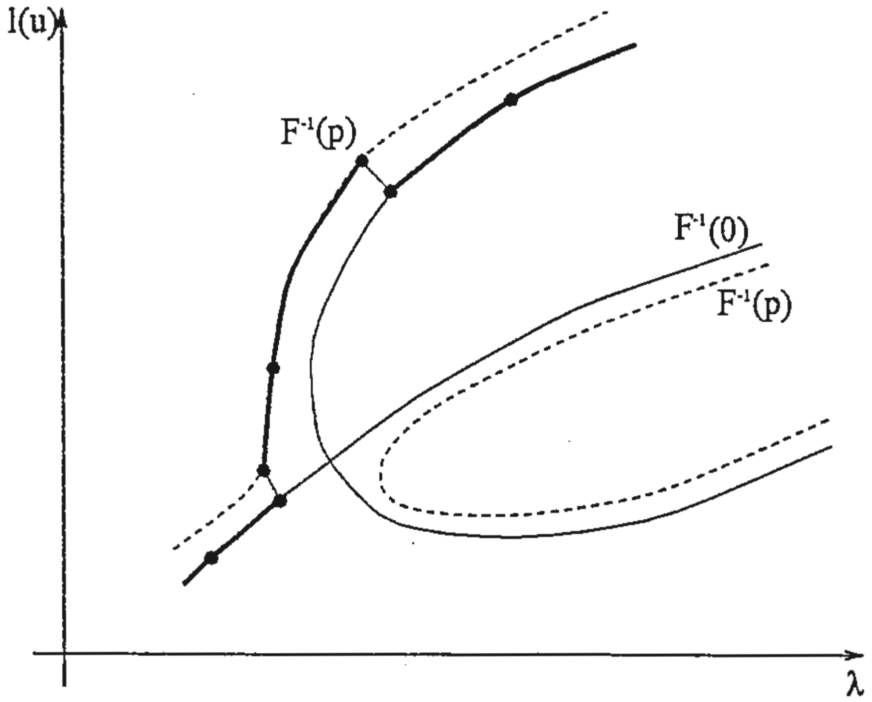

(u_h^4, λ_h^4)
(u_H^4, λ_H^4)
(u_h^3, λ_h^3)
(u_h^2, λ_h^2)
(u_H^3, λ_H^3)
(u_h^1, λ_h^1)
(u_H^2, λ_H^2)
(u_H^1, λ_H^1)

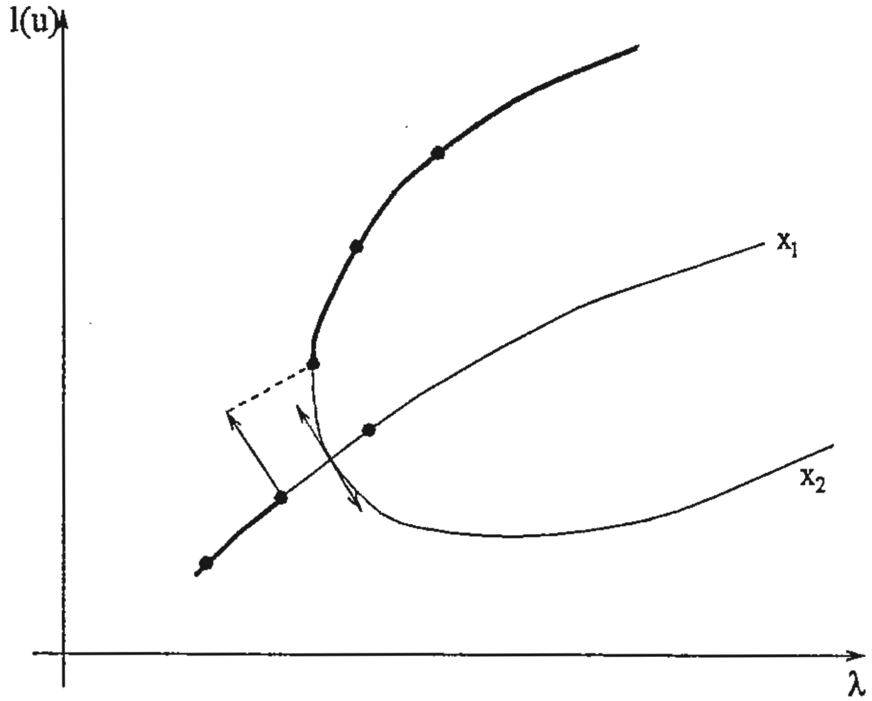

B_H
(u_h^4, λ_h^4)
(u_h^3, λ_h^3)
(u_H^4, λ_H^4)
(u_h^1, λ_h^1)
(u_H^1, λ_H^1)


l(u)
λ


l(u)
λ


l(u)
λ


l(u)
$F^{-1}(p)$
$F^{-1}(0)$
$F^{-1}(p)$
λ


l(u)
x_1
x_2
λ

... AND CONCERNING STABILITY

Let a formal dynamical system:

$$(*)\left|\begin{array}{l}\dot{y} = f(y) \\ y(0) = y_0\end{array}\right. \qquad \underline{y = y(t)\ ,\ t \in [0, +\infty)}$$

Then, what about:

$$(**)\left|\begin{array}{l}\dot{y} = f(y) \\ y(0) = \breve{y}_0 \\ \|\breve{y}_0 - y_0\| = \varepsilon\ ,\end{array}\right.$$

specially as $t \longrightarrow +\infty$

Or:

$$(***)\left|\begin{array}{l}\dot{y} = f(y) + \varepsilon\, g(y) \\ y(0) = y_0\end{array}\right.$$

$(*) \neq (**)$ Lyapunov stability

$(*) \neq (***)$ Structural stability or "imperfection sensitivity"

A SIMPLE STABILITY RESULT

EQUATIONS OF MECHANICS

$$\ddot{x} = f(x, \dot{x})$$

$$\hookrightarrow \begin{cases} \dot{x} = y \\ \dot{y} = f(x, y) \\ + \text{I.D.} \end{cases}$$

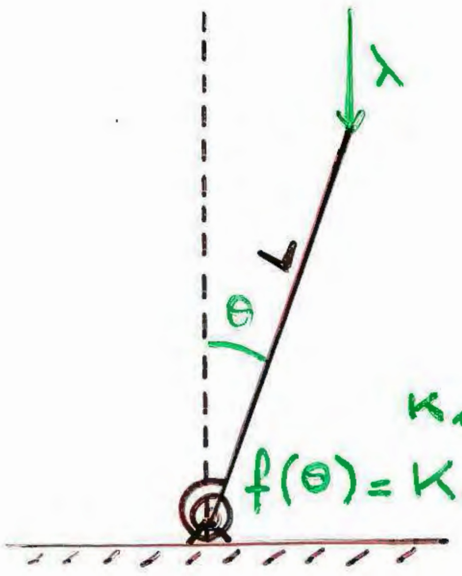
$$\hookrightarrow \|\hat{x}(0) - x_0\| < \eta_1, \ \|\hat{y}(0) - y_0\| < \eta_2 \ \cdots$$

THEOREM (Lejeune - Dirichlet)

Let a finite dimensional conservative system, $q(t) \in \mathbb{R}^n$, with a potential energy $E(q)$.

Then,

an equilibrium state q_e is stable if it achieves a strict local minimum of $E(q)$.



$$\boxed{\lambda L \sin\theta = f(\theta)}$$

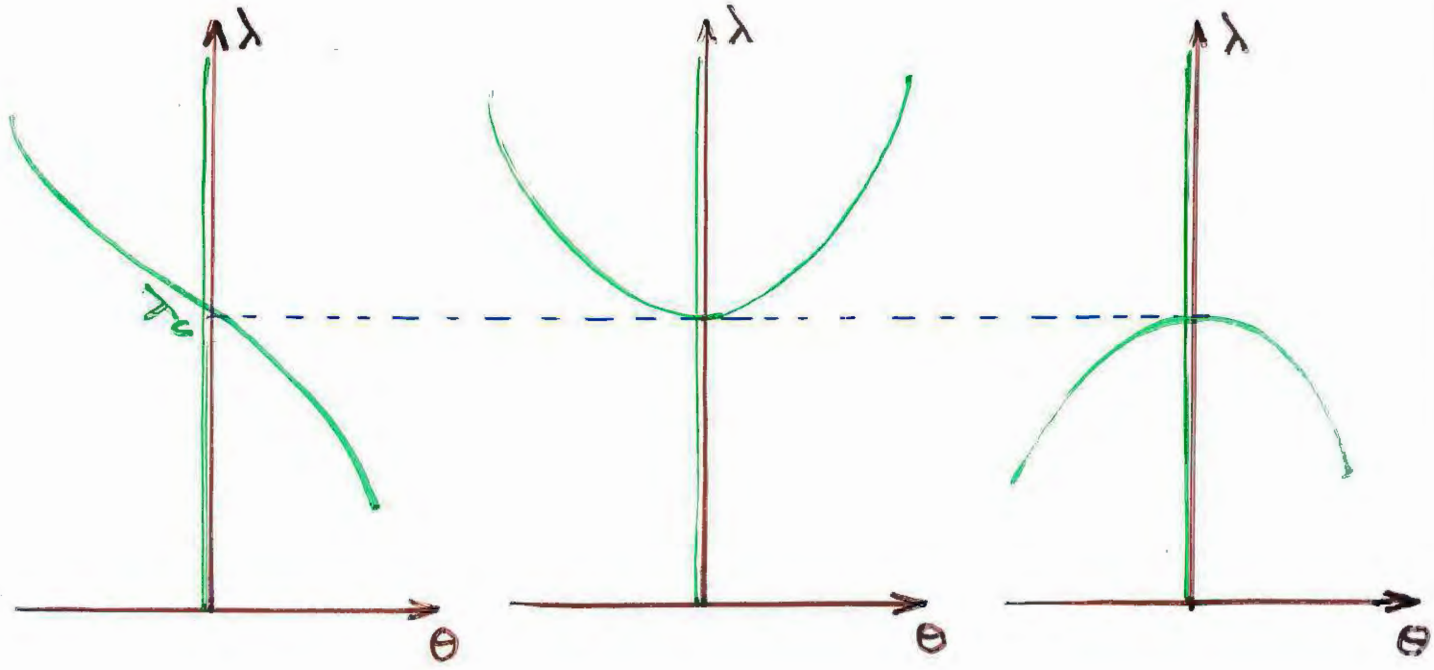
$$K_1 > 0$$

$$f(\theta) = K_1\theta + K_2\theta^2 + K_3\theta^3 + \cdots$$

Deux branches de solutions :

- $\left\{ (\theta,\lambda) \,/\, \theta = 0,\ \lambda \in \mathbb{R}^+ \right\}$

- $\left\{ (\theta,\lambda) \,/\, \lambda = \dfrac{f(\theta)}{L\sin\theta} \right\}$



$$E(\theta, \lambda) = \int_0^\theta f(s)\, ds + \lambda L \cos\theta$$

$$\Rightarrow \; -\frac{\partial E}{\partial \theta} = 0 = f(\theta) - \lambda L \sin\theta$$

$$-\frac{\partial^2 E}{\partial \theta^2} = f'(\theta) - \lambda L \cos\theta$$

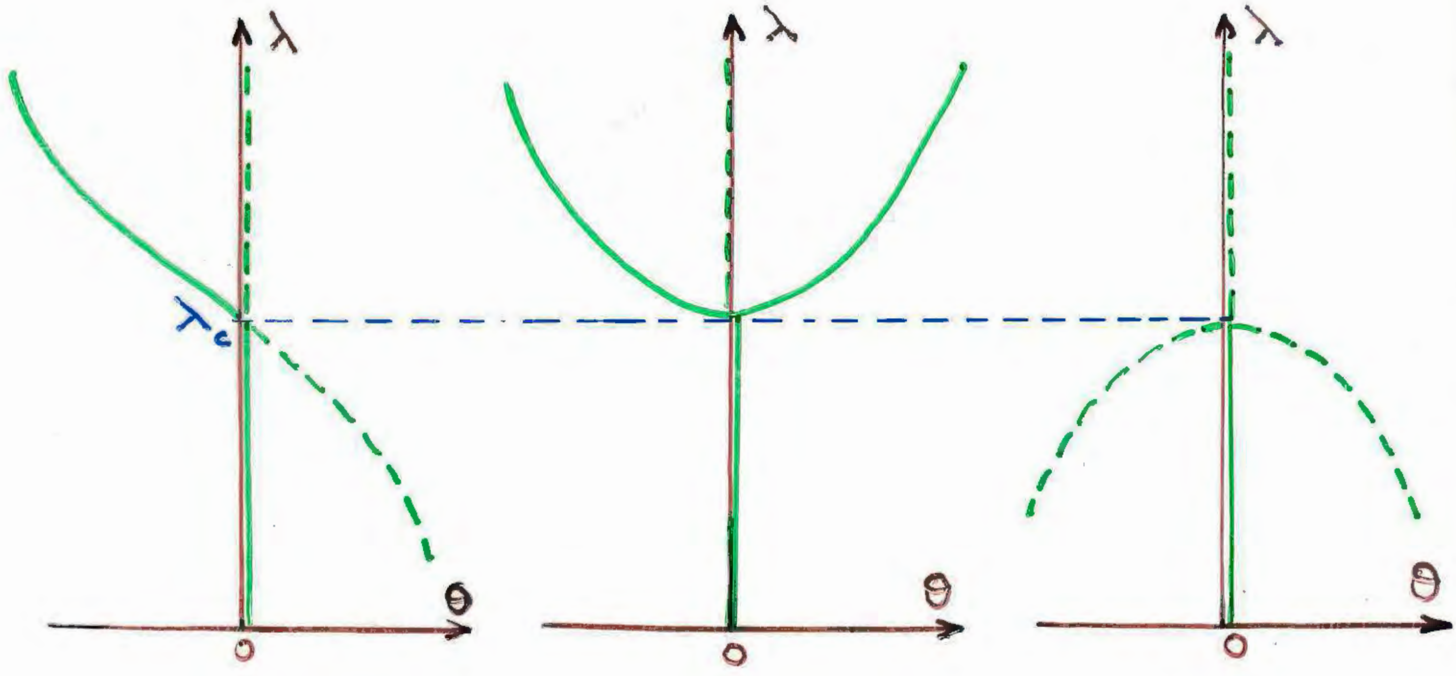
Sur la branche triviale :

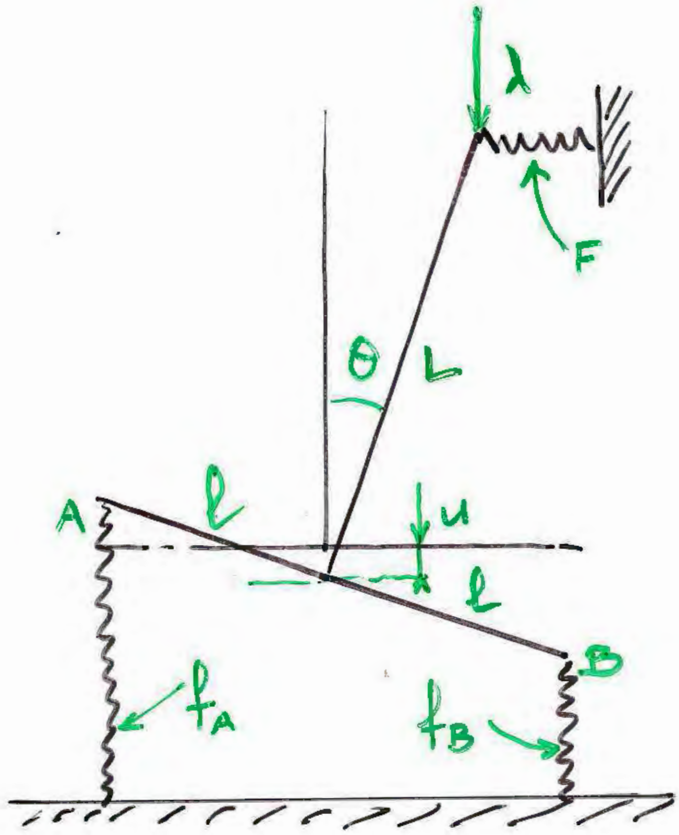
$$\lambda < \lambda_c \implies E_{,\theta\theta} > 0$$

$$\lambda = \lambda_c \implies E_{,\theta\theta} = 0$$

$$\lambda > \lambda_c \implies E_{,\theta\theta} < 0$$

Sur la branche bifurquée :





$$F = -(kd + md^2 + nd^3)$$

$$d = L\theta$$

$$d_A = u - \ell\theta$$

$$d_B = u + \ell\theta$$

$$\mathcal{E}(u,\theta,\lambda) = \frac{E}{2}\left[(u+\ell\theta)^2 + (u-\ell\theta)^2\right] + \frac{k}{2}(L\theta)^2 + \frac{m}{3}(L\theta)^3 + \frac{n}{4}(L\theta)^4 - \lambda\left(u + \frac{L\theta^2}{2}\right)$$

Equilibrium :

$$\begin{cases} \mathcal{E}_{,u} = 0 : & 2Eu - \lambda = 0 \\ \mathcal{E}_{,\theta} = 0 : & (2E\ell^2 + kL^2)\theta + mL^3\theta^2 + nL^4\theta^3 - \lambda L\theta = 0 \end{cases}$$

$$(u,\theta) = \left\{\left(\frac{\lambda}{2E}, 0\right), \forall\lambda\right\}$$

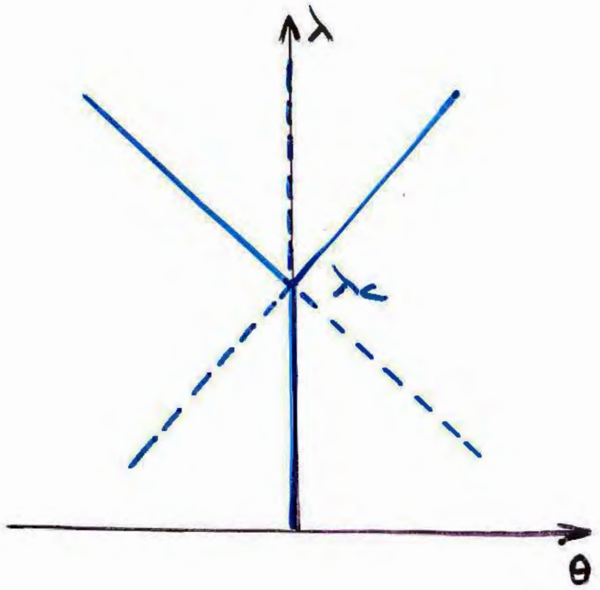
$$\lambda_c = \frac{2E\ell^2 + kL^2}{L}$$

$$\begin{pmatrix} \mathcal{E}_{,uu} & \mathcal{E}_{,u\theta} \\ \mathcal{E}_{,u\theta} & \mathcal{E}_{,\theta\theta} \end{pmatrix} = \begin{pmatrix} 2E & 0 \\ 0 & (\lambda-\lambda_c)L + 2mL^3\theta + 3nL^4\theta^2 \end{pmatrix}$$

$m \neq 0$

$n = 0$

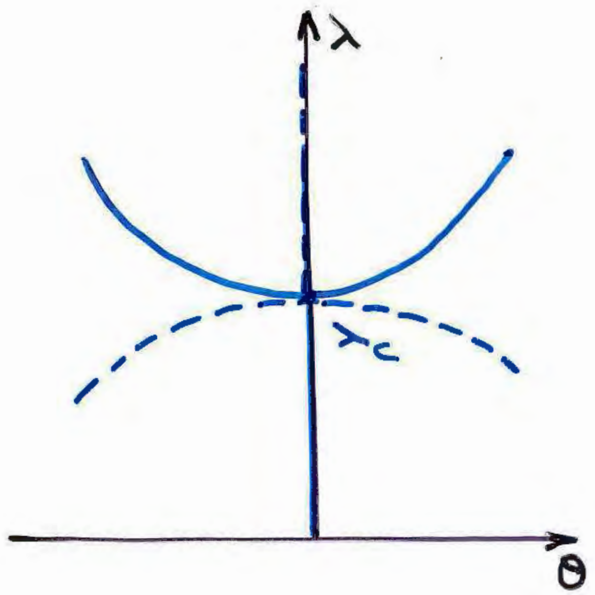
$$\begin{pmatrix} 2E & 0 \\ 0 & 2mL^3\theta \end{pmatrix}$$



$m = 0$

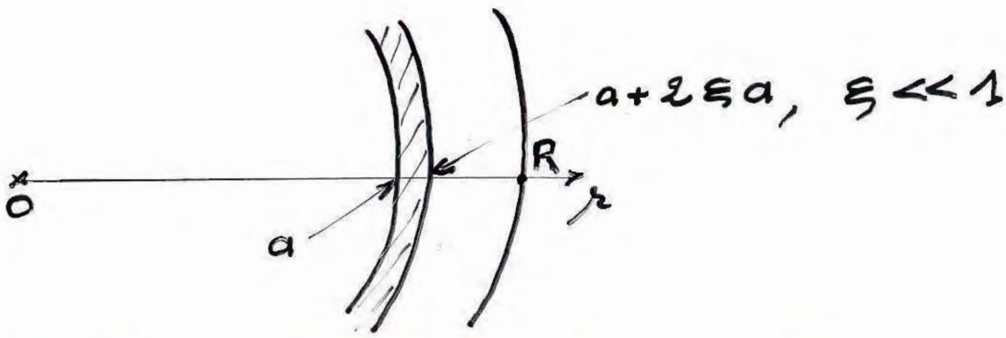
$n \neq 0$

$$\begin{pmatrix} 2E & 0 \\ 0 & 3nL^4\theta^2 \end{pmatrix}$$



A COUNTER EXAMPLE OF ENERGY CRITERION FOR INFINITE DIMESIONAL SYSTEMS

A.E. Green & R.T. Shield



$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} + \frac{F}{c^2}$$

$$c^2 = (\lambda + 2\mu)/\rho$$

$$u = \frac{1}{r}\Big[f(r+ct) - f(r-ct)\Big] - \frac{1}{r}\Big[f'(r+ct) - f'(r-ct)\Big]$$

$$\lim_{r\to 0} u = 0 \quad , \quad \lim_{r \to 0} \frac{u}{r} = -\frac{2}{3} f'''(ct)$$

$$u_0 = u(t=0) = 0 \; ; \; u_1 = \frac{\partial u(t=0)}{\partial t} = \frac{2c}{r}\left[\frac{1}{r} f'(r) - f''(r)\right]$$

$$\begin{cases} f(r) = 0 & : 0 \leq r \leq a \\ f(r) = \frac{1}{\xi^5 a^5}(r-a)^4(r-a-2a\xi)^4 & : a \leq r \leq a + 2\xi a \\ f(r) = 0 & : a + 2\xi a \leq r \end{cases}$$

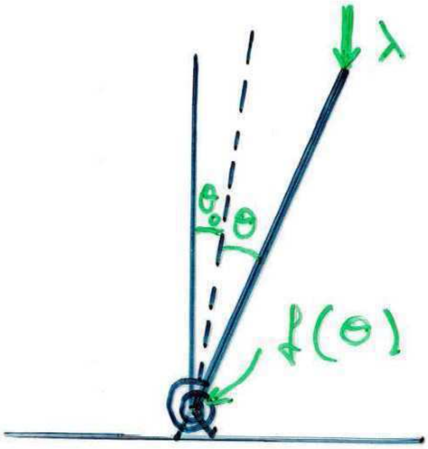
$\Rightarrow$ u_0 and u_1 are both zero as $\xi \to 0$

$\Rightarrow$ At $t = a/c$, for $0 \leq r \leq 2\xi a$

$$u = \frac{1}{\xi^5 a^5} r^2 (2\xi a - r)^3 (7r - 6\xi a)$$

$$\left(\frac{u}{r}\right)_{r=\xi a} = 1 \text{ and } \left(\frac{\partial u}{\partial r}\right)_{r = \xi a} = 6 \; !$$

IMPERFECTION SENSITIVITY



$$f(\theta) = K_1\theta + K_2\theta^2 + K_3\theta^3 + \dots$$

$$K_1 > 0$$

$$f(\theta) - \lambda L \sin(\theta + \theta_0) = 0$$

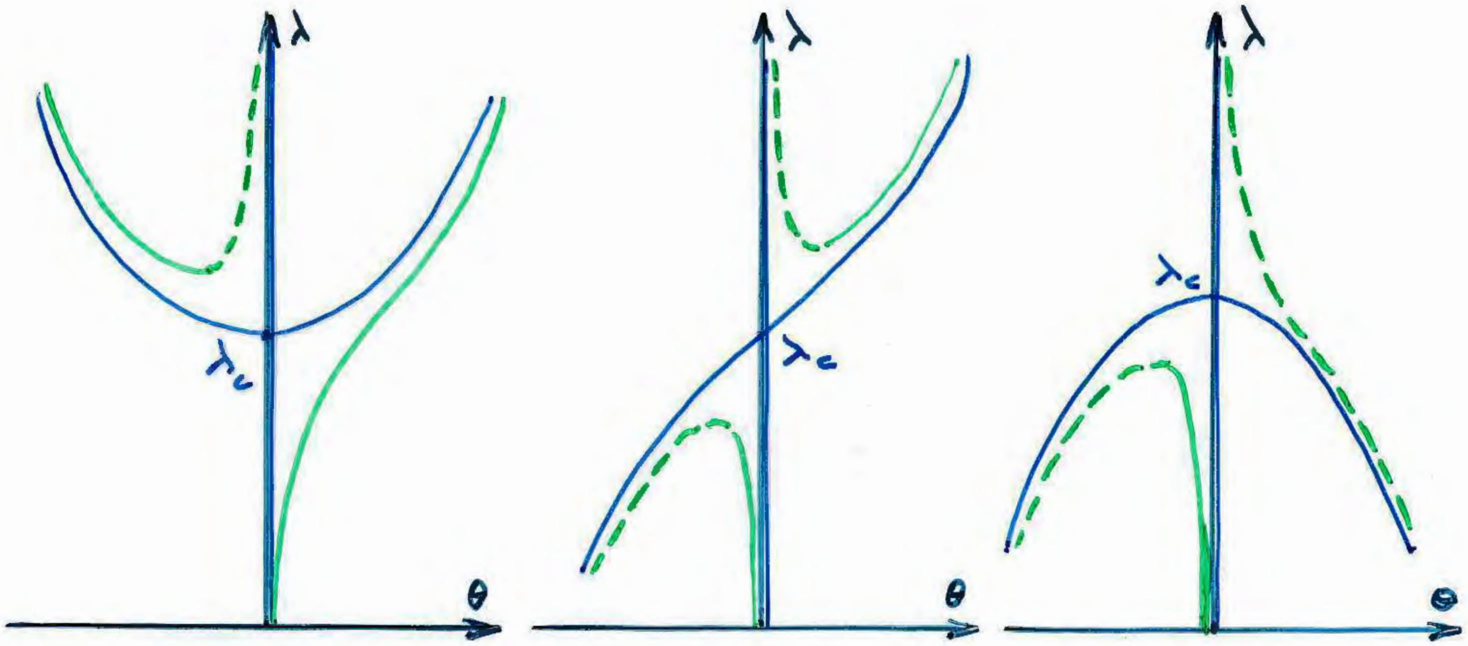
$\Rightarrow$ either $(\lambda - \lambda_c)\theta - \lambda_1\theta^2 = \lambda_c\theta_0$, $K_2 \neq 0$

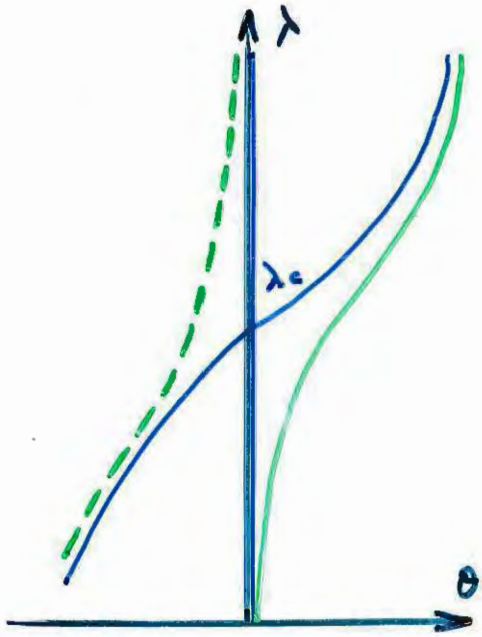
or $(\lambda - \lambda_c)\theta - \lambda_2\theta^3 = \lambda_c\theta_0$, $K_2 = 0$

1) $\{(0,\lambda),\ \lambda \in \mathbb{R}^+\}$ is not a solution

2) There is no bifurcation point

3) Non connected branches appear.





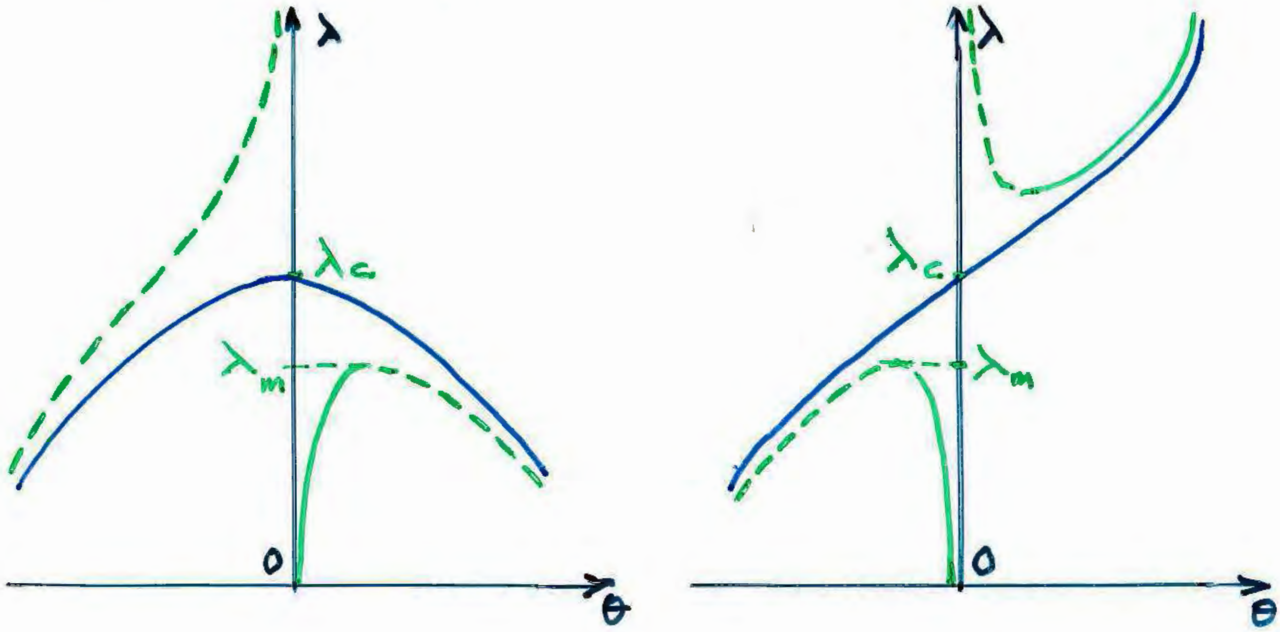
In addition to the discussion $K_2 = 0$,

$K_2 > 0$,

!

$\theta_0 > 0$ or $\theta_0 < 0$

$\Rightarrow$ Maximal load reduction / Sensitivity



$$\frac{d\lambda}{d\theta} = 0$$

$$\lambda_c - \lambda_m = 3\left(\lambda_c \frac{|\theta_0|}{2}\right)^{2/3}\left(|\lambda_2|\right)^{1/3}$$

or

$$\lambda_c - \lambda_m = 2\left(\lambda_c |\theta_0| |\lambda_2|\right)^{1/2}$$

In general; let a_0 an imperfection parameter, Then, for sensitive structures having some symmetries, $\lambda_c - \lambda_m \propto a_0^{2/3}$

$$E(u,\lambda) \longrightarrow E(u,\lambda,\delta)$$

Assumptions

* $(0,\lambda)$ is a solution without imperfection

$$\hookrightarrow \quad E_{,u}(0,\lambda,0) = 0$$

* $\exists$ nontrivial solution for small δ

* $E_{,uu}(0,\lambda_c,0)\,U = 0$

* $E(u,\lambda,\delta) = E(u,\lambda) - \delta D(u) + \text{h.o.t}$

$$\left\{\begin{array}{l}\text{Then Find } u = \xi U + w^* \\ \text{with } \xi \in \mathbb{R},\ U \in \mathcal{N},\ w^* \in \mathcal{N}^\perp\end{array}\right.$$

$$\exists!\ w^*(\xi,\lambda,\delta) = \hat{w}(\xi,\lambda) + \mathcal{O}(\delta)$$

$$\xi^2 \hat{w}_2(\lambda)$$

$$\lambda - \lambda_c = \delta \frac{D(U)}{C_1 \xi} + C_2 \xi^2.$$

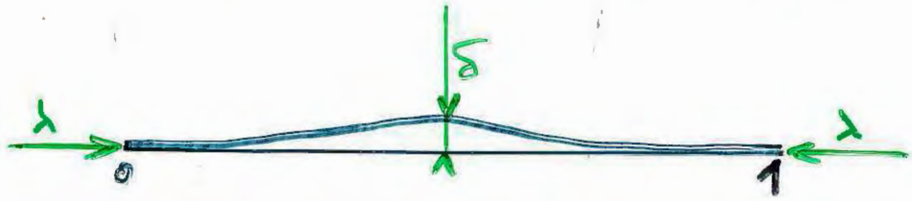
Imperfection sensitive structure.

$$\lambda_c - \lambda_m = \left\{ \frac{D(U)}{4 C_1^2 |C_2|} \right\}^{1/3} \delta^{2/3}$$

The imperfection may appear as a forcing

e.g. $\begin{cases} \theta'' + \lambda \sin\theta = \delta f(x) \\ \theta'(0) = \theta'(1) = 0, \end{cases}$

either as an initial deflection:



or an additional force:

Or as a singular perturbation:

$$\begin{cases} L_1(w,\lambda) + \delta L_2(w) = 0 \\ + \\ \text{B.C.} \end{cases}$$

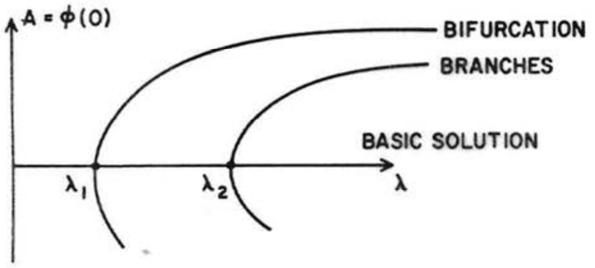


Figure 1.1

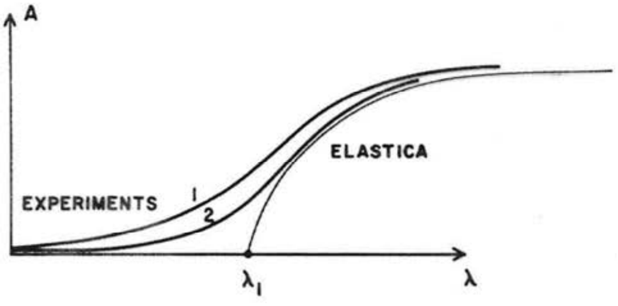


Figure 1.2

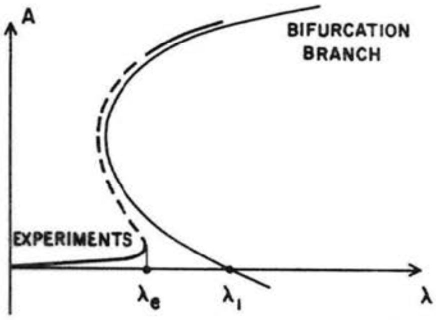


Figure 1.3

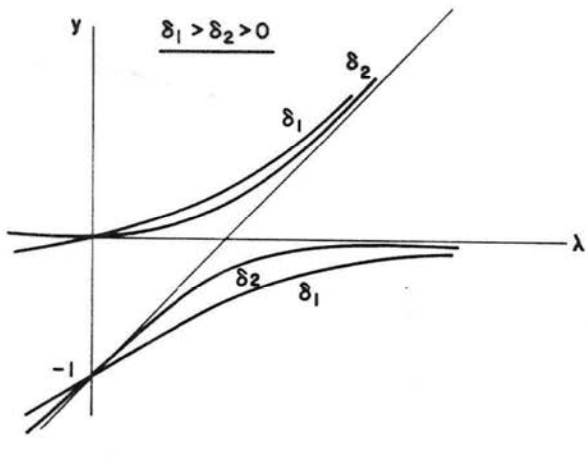


Figure 2.1a

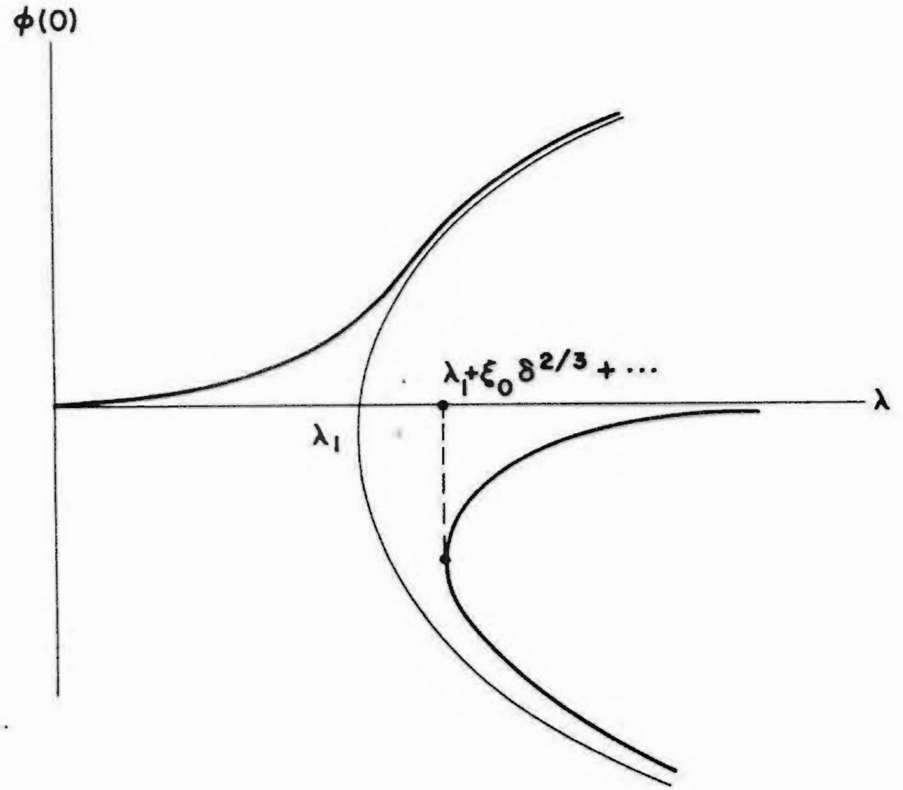


Figure 3.2

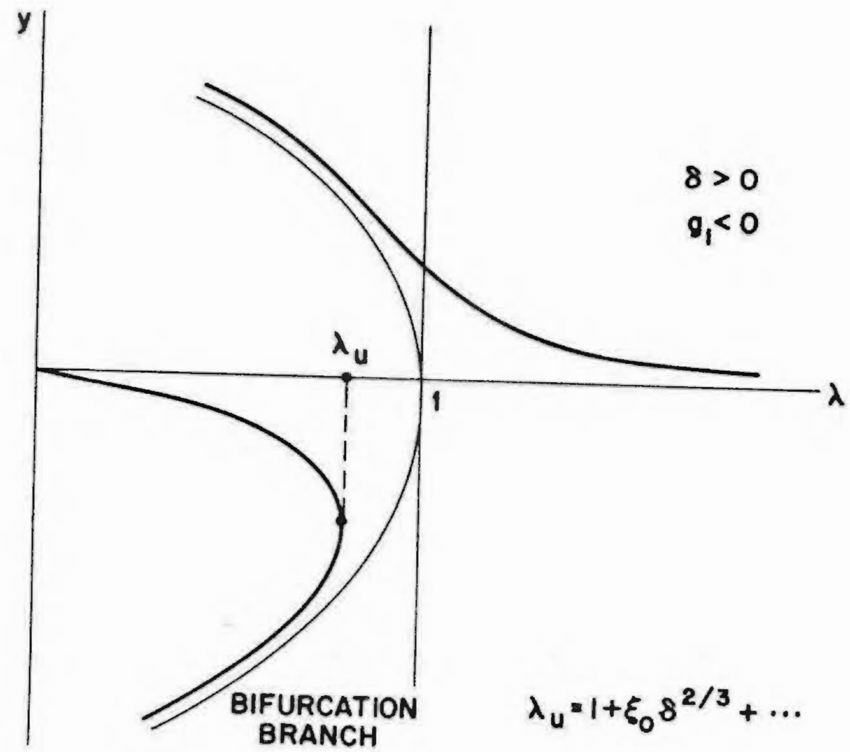


Figure 5.1

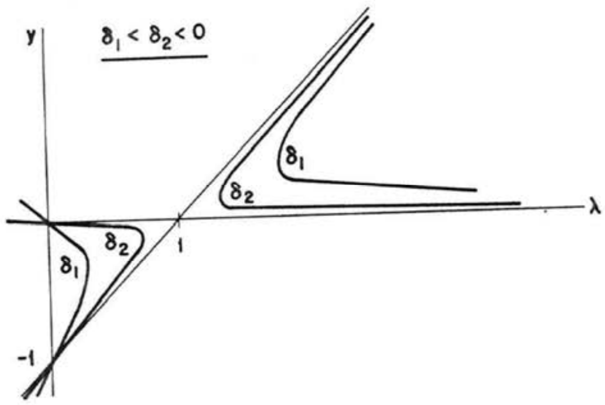


Figure 2.1b

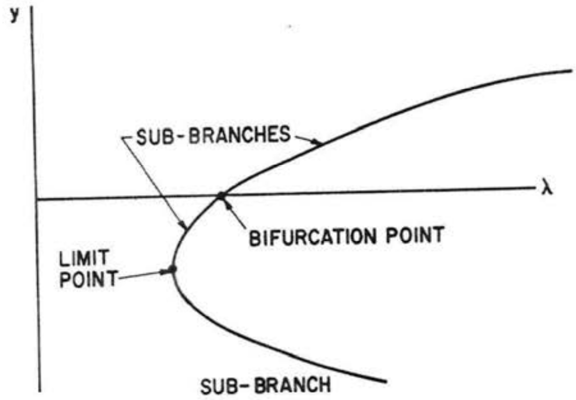


Figure 2.2

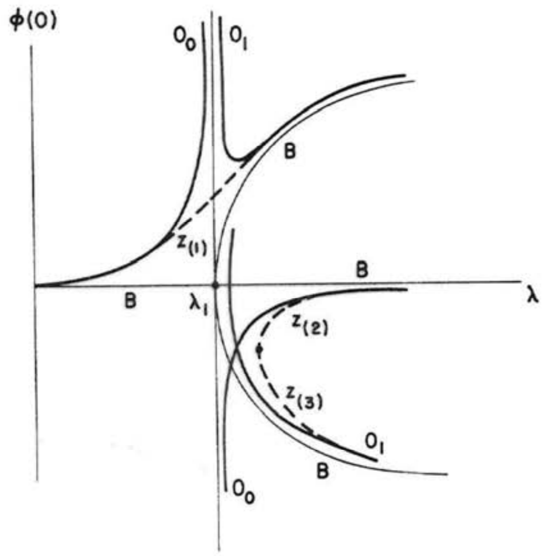
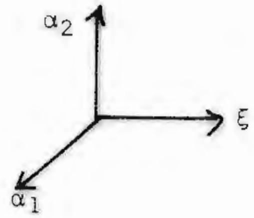
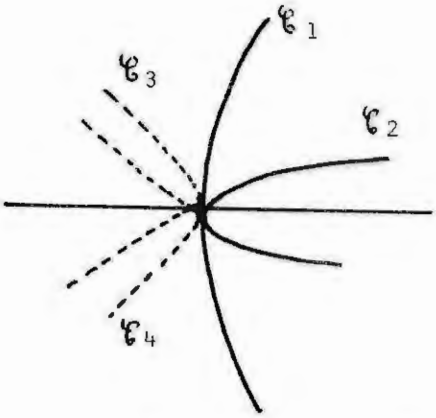
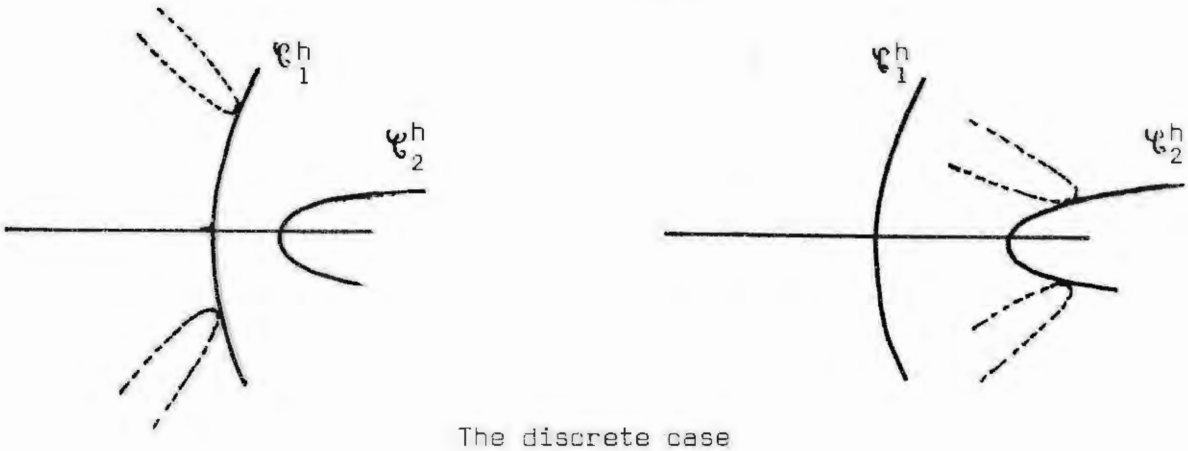


Figure 3.1



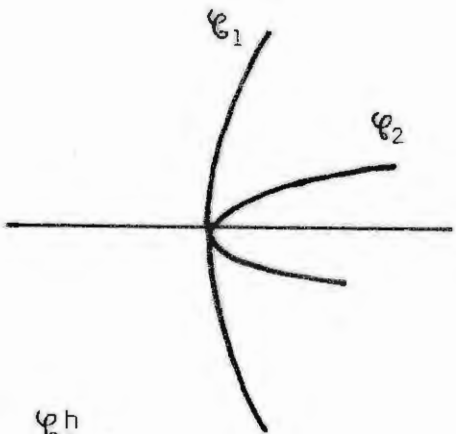


The continuous case

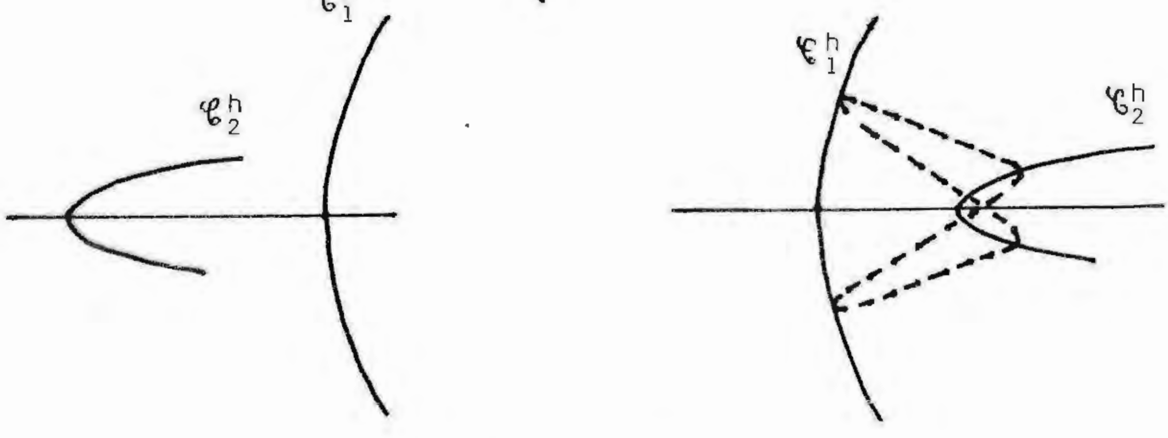


The discrete case

Figure 5.3



The continuous case



The discrete case

Figure 5.4

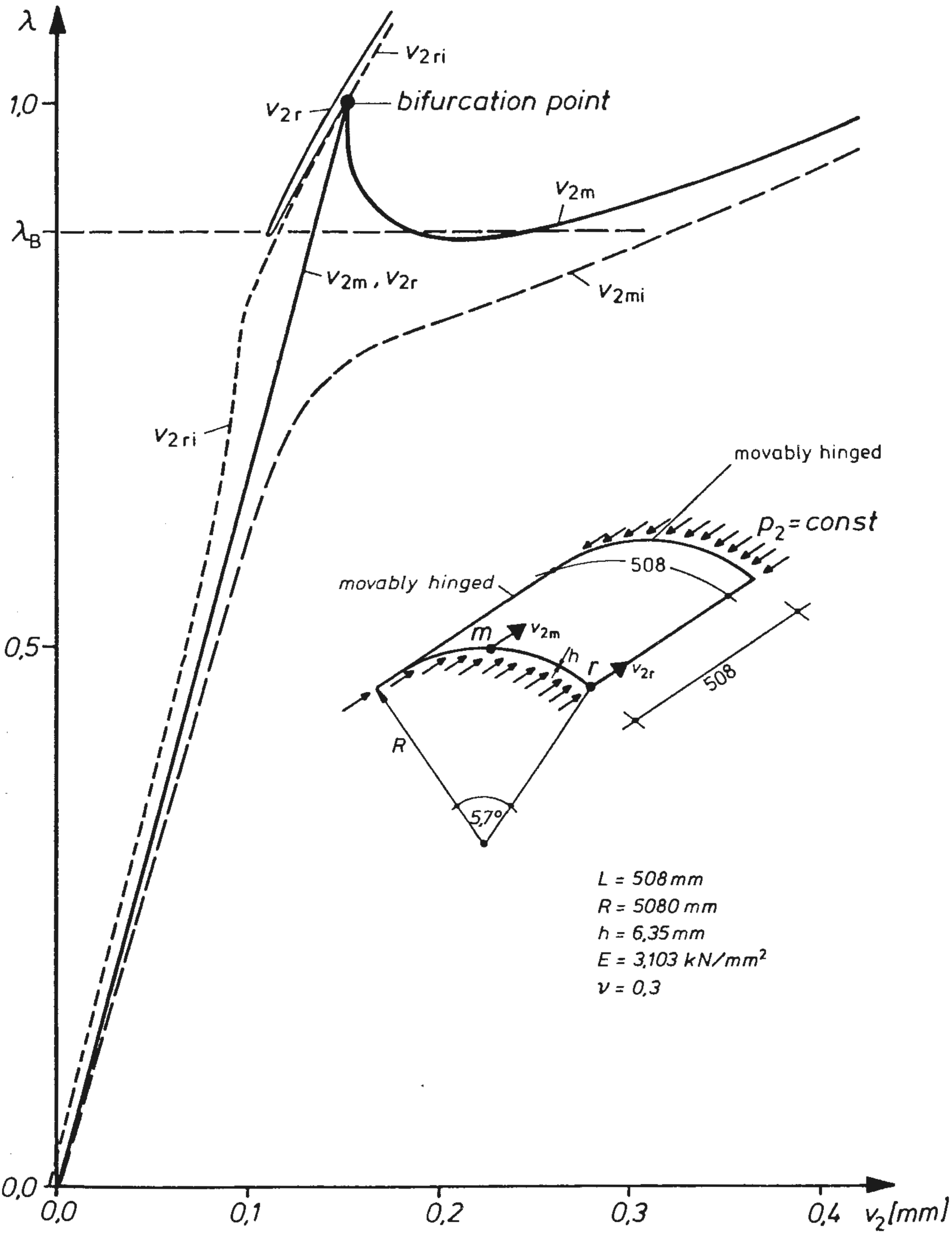
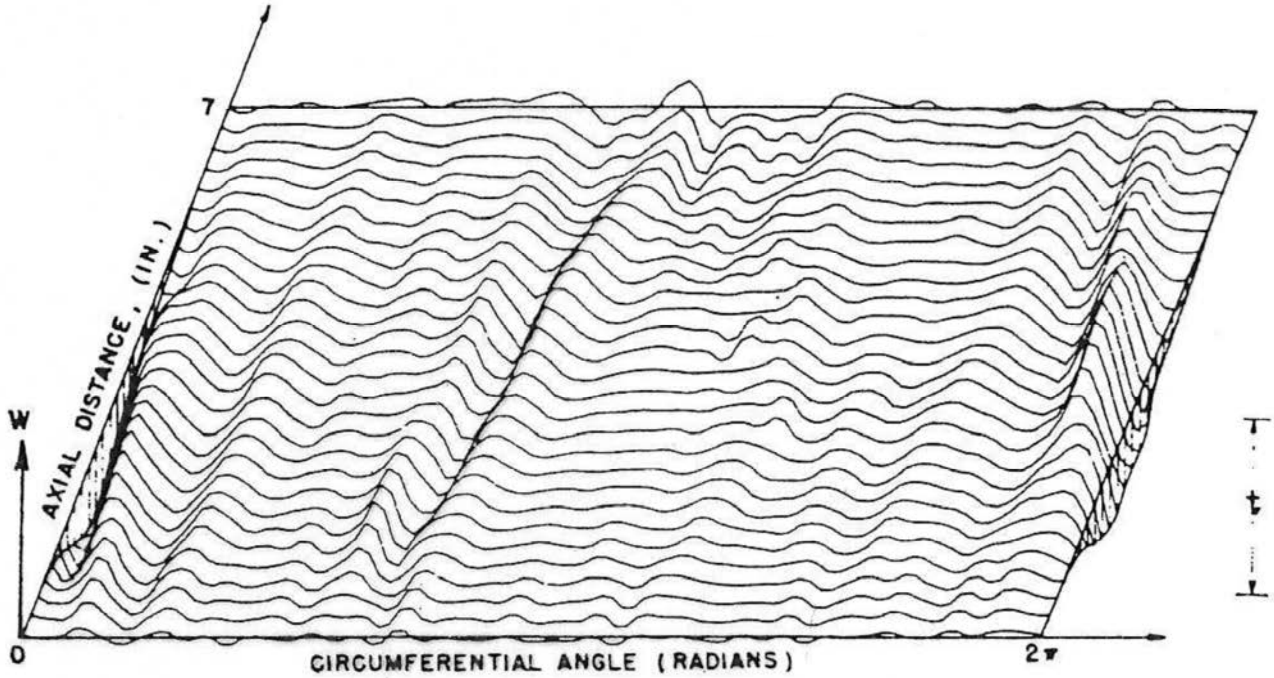
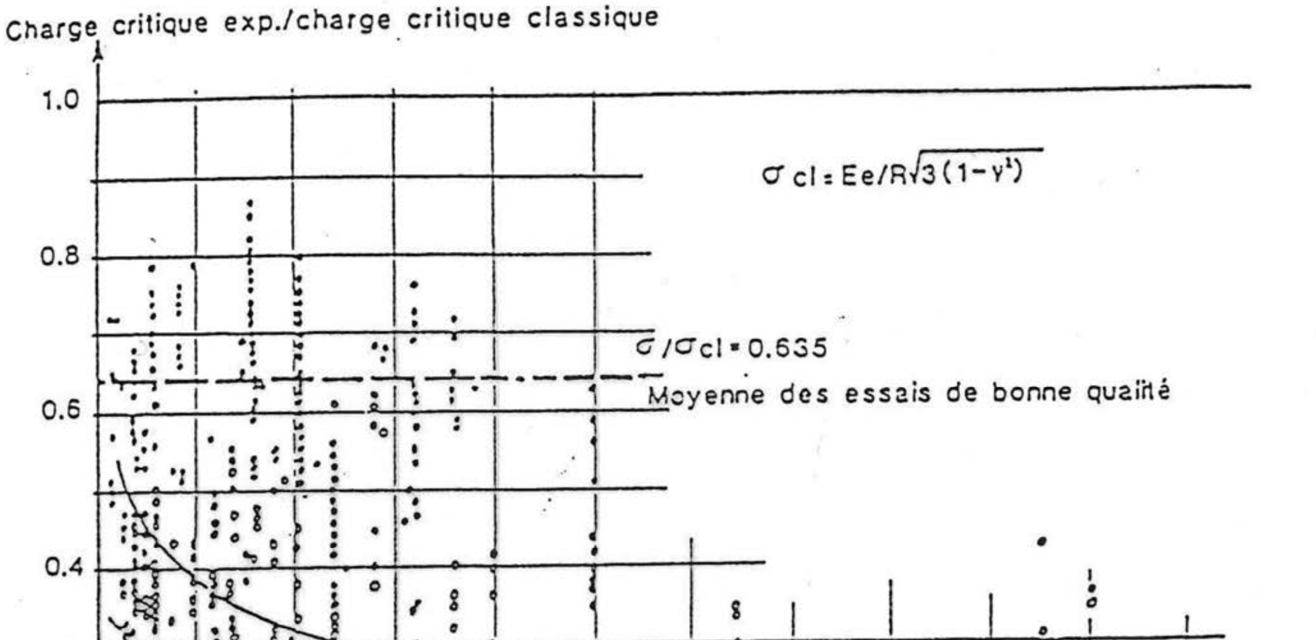
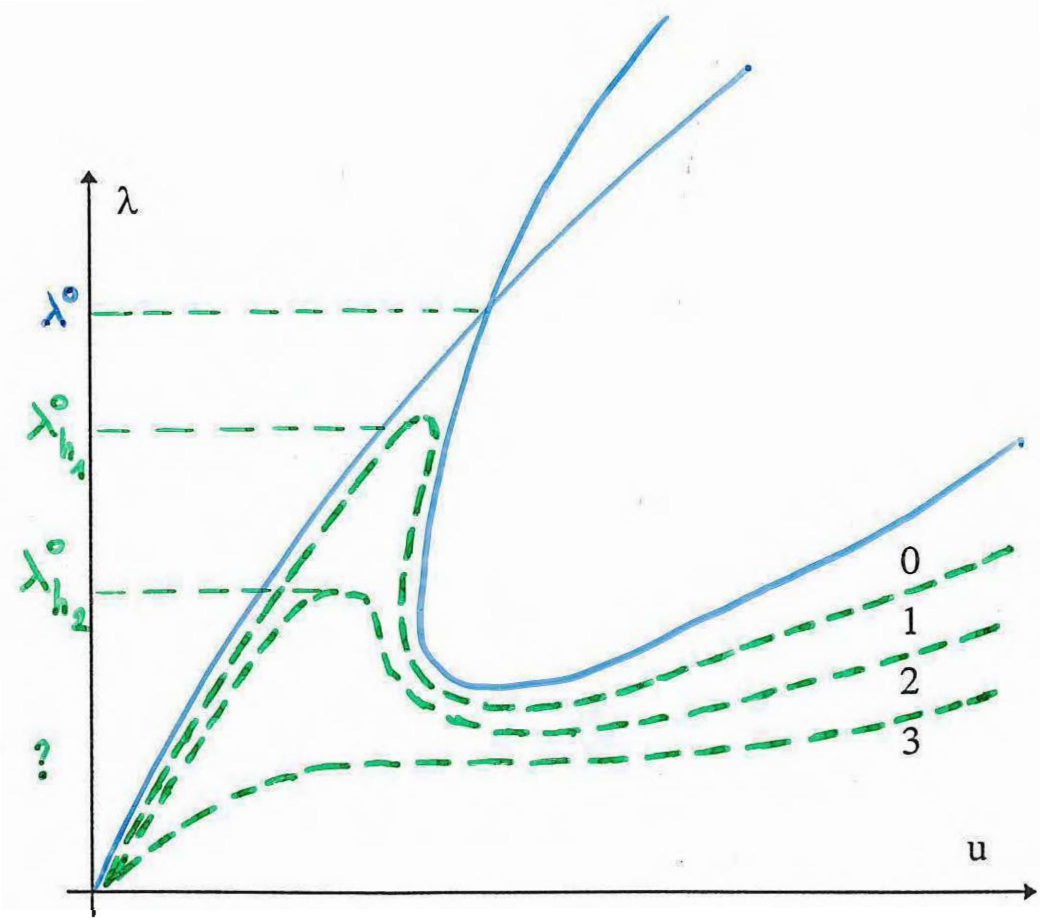


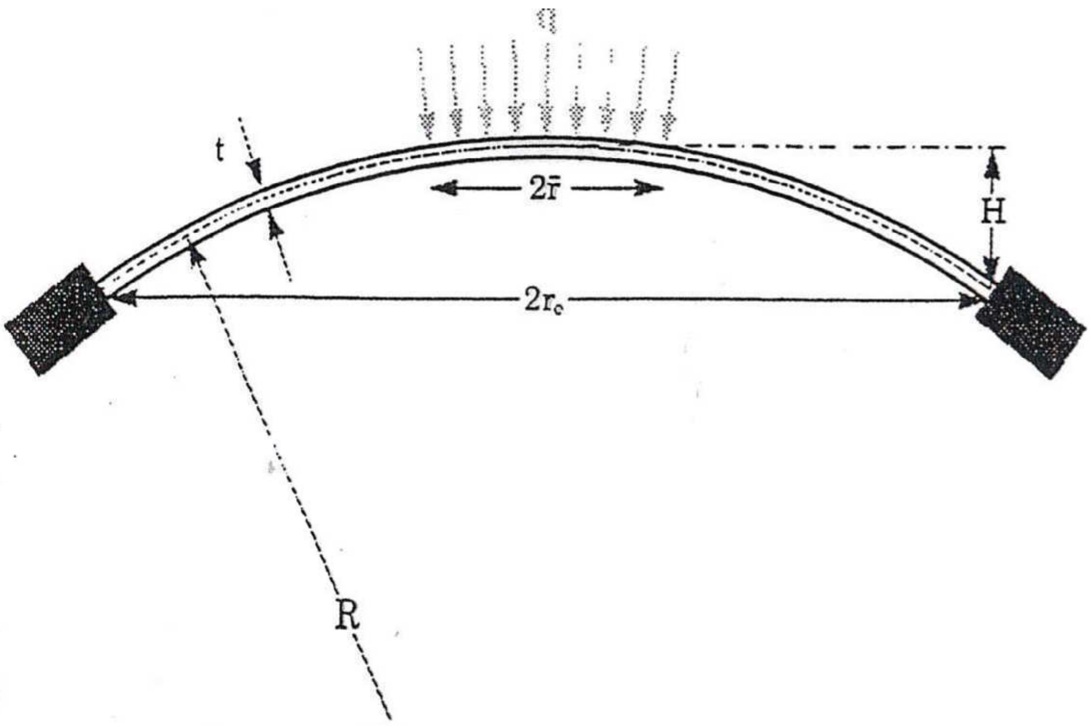
Fig. 14: Bifurcation of axially loaded shallow cylindrical shell

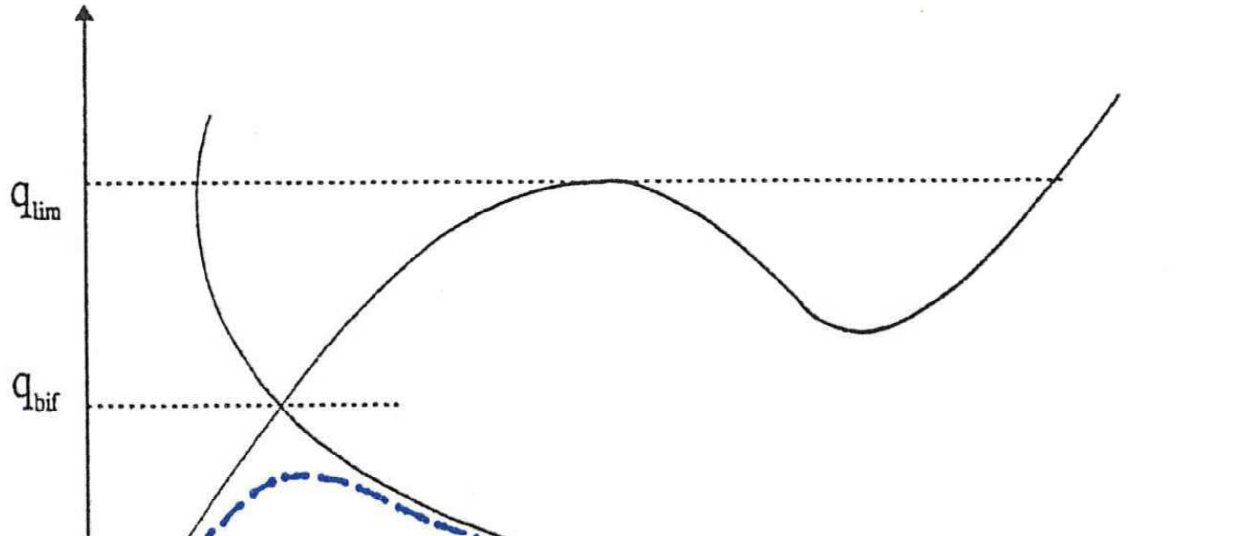


(b) SHELL A-9 AT P/P_{cl} = 0.722




λ
λ^0
$\lambda^0_{h_1}$
$\lambda^0_{h_2}$
?
0
1
2
3
u


q
t
2r̄
H
2r_c
R


q_lim
q_bif
||w||