MULTIPLICITY OF EQUILIBRIUM STATES OF NONLINEAR SHELLS AND MEMBRANES

- **INTRODUCTION**
- **THE CASE OF A SPHERICAL CAP**
- **THE GENERAL CASE OF A SHALLOW SHELL**

→ Let $\mathbb{H}$ be a Hilbert space, and:

$$(*) \quad Lu - \lambda u + C(u) = 0$$

L linear.

→ Let $\mathcal{B} = \{(u,\lambda) \in D_L \times \mathbb{R} / u \text{ sol. of } * \text{ for } \lambda, u \neq 0\}$

THEN:

$(0,\mu) \in \overline{\mathcal{B}}$ is a bifurcation point of $(*)$ $\Rightarrow \mu \in \sigma(L)$ if:

1) $C(u)$ continuous : $D_L \longrightarrow \mathbb{H}$
2) $\|C(u)\| / |||u||| \xrightarrow[|||u||| \to 0]{} 0$

RECALL:

$$\sigma = \sigma_p \cup \sigma_c \cup \sigma_r$$

$$\equiv \sigma_d \cup \sigma_{ess}$$

(σ_d = {isolated eigenvalues of finite multiplicity}).

USUALLY:

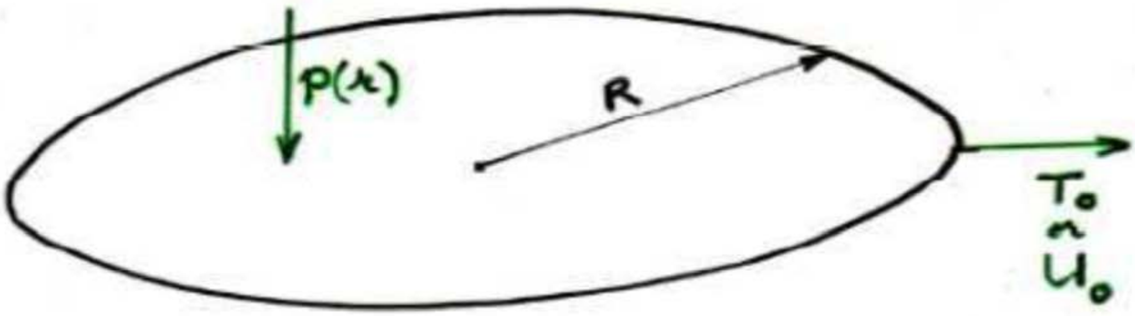
$\mu \in \sigma_d$. Then sufficient conditions for $(0,\mu)$ to be a bifurcation point.

THEN:

What about $(0,\mu)$ if $\mu \in \sigma_{ess}$?

BUCKLING OF A FLAT CIRCULAR MEMBRANE

(A.J. Callegari, E.L. Reiss, H.B. Keller)



$$x = \frac{r}{R} \quad , \quad s(x) = \frac{\sigma_r(r)}{E P^{1/3}}$$

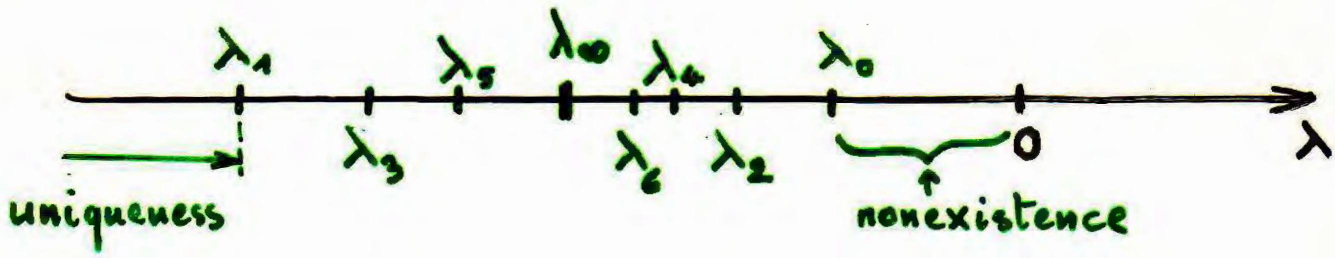
$$Ls \equiv (x^3 s')' + \frac{G(x)}{s^2} = 0$$

$$\left(p(r) = C^{ste} \Rightarrow G(x) = x^3 \right)$$

- $s'(0) = 0$
- $\begin{cases} s(1) = \lambda \equiv \dfrac{T_0}{EP^{1/3}} \\ \text{or} \\ s'(1) + (1-\nu) = u \equiv \dfrac{u_0}{RP^{1/3}} \end{cases}$

* $\lambda > 0$: unique solution
* $U = 0$: unique solution.

↳ Let's look at $\lambda < 0$ (or $U < 0$)



$$\{\lambda_{2n}\} \longrightarrow \lambda_\infty$$

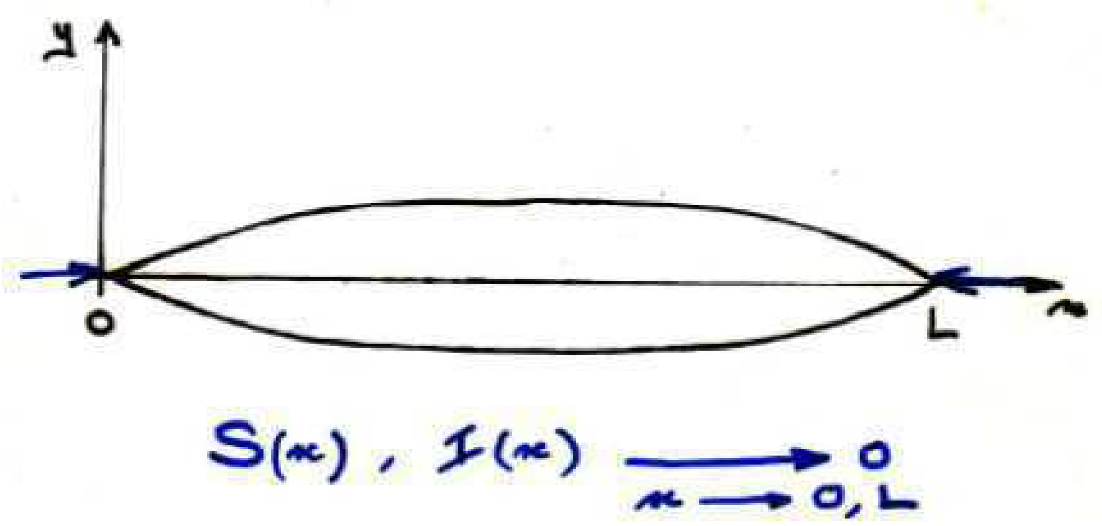
$$\{\lambda_{2n+1}\} \longrightarrow \lambda_\infty$$

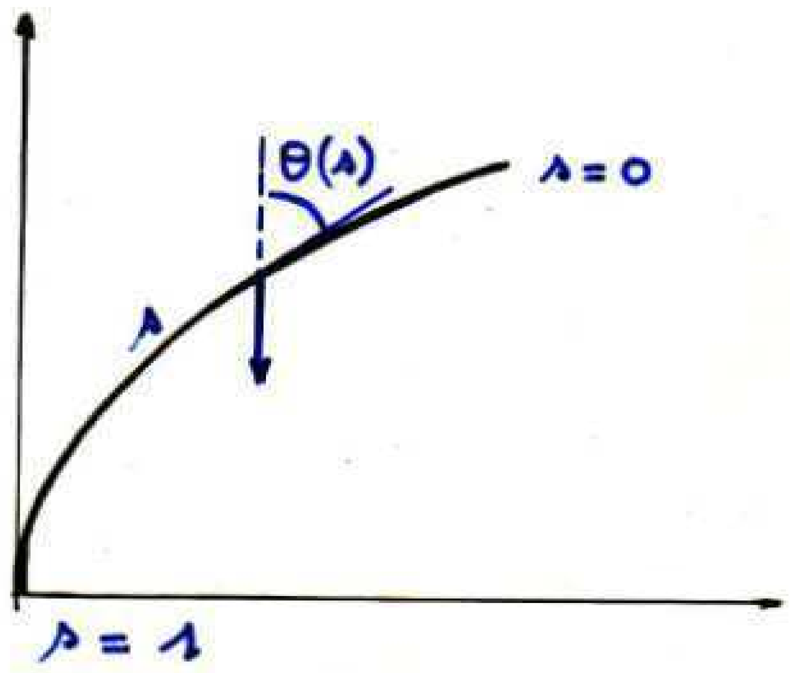
such that :

* $\lambda \in \,]\lambda_{2n}, \lambda_{2n-2}[$: exactly $2n$ solutions
* $\lambda \in \,]\lambda_{2n-1}, \lambda_{2n+1}[$: exactly $2n+1$ solutions
* $\lambda = \lambda_\infty$: infinitely many solutions

BUCKLING OF A TAPERED ELASTICA

(Ch. Stuart)





$$\begin{cases} u(s) \equiv \theta(s) \\ A(s) \equiv EI(s) \end{cases}$$

$$\{A(s)\, u'(s)\}' + \mu \sin u(s) = 0 \;, \quad s \in (0,1)$$

$$u(1) = \lim_{s \to 0} A(s)\, u'(s) = 0$$

$$\int_0^1 A(s)\, u'(s)^2\, ds < +\infty$$

avec $\displaystyle \lim_{s \to 0} \frac{A(s)}{s^p} = L$

↳ Linearized problem

idem but :

$$\{A(s)\, u'(s)\}' + \mu\, u(s) = 0 \;, \quad s \in (0,1]$$

$$\boxed{\lim_{s \to 0} \frac{A(s)}{s^p} = L}$$

→ $0 \leqslant p < 2$: * $\exists \, \Lambda(A)$ s.t. problem $(\mathcal{P})$ has nontrivial solutions for $\mu \geqslant \Lambda(A)$.

* Almost usual beam buckling.

→ Very strong changes at $p = 2$:

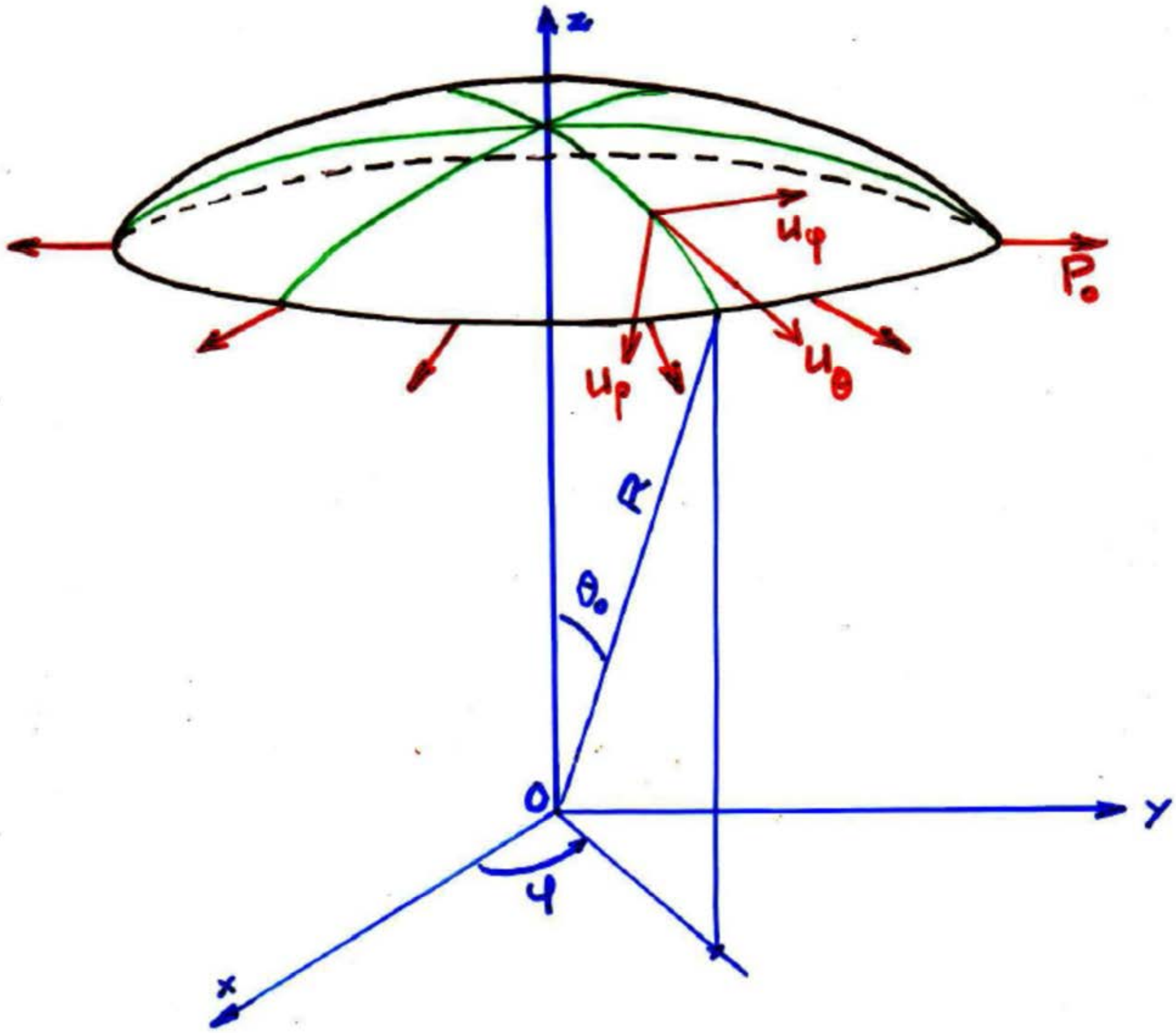
$\exists \, \Lambda_e \left(= \frac{L}{4}\right)$ s.t., $\forall \mu > \Lambda_e$

* Infinitely many distinct solutions

* Infinitely many zeros in $(0,1)$

* Very steep boundary layers

SPHERICAL CAPS



- Thin shell
- Shallow shell
- Axisymmetry

$$R-h \leq \rho \leq R+h \; ; \quad 0 \leq \theta \leq \theta_0 \; ; \quad 0 \leq \varphi \leq 2\pi$$

$$\frac{h}{R\theta_0^2} \ll 1 \; ; \quad \theta_0^2 \ll 1$$

THE MODEL

* Displacements:

$$\begin{cases} u_\varphi = 0 \\ u_\rho = R\theta_0\, w(\theta_0 \eta) \\ u_\theta = R\theta_0\, u(\theta_0\eta) + (\rho - R)\, V(\theta_0 \eta) \end{cases}$$

where:

$$\begin{cases} \theta = \theta_0 \eta\,, \qquad \eta \in [0,1] \\ V = \theta_0 u + w'\,, \quad (\cdot)' = \dfrac{\partial}{\partial \eta} \end{cases}$$

* Nonlinear strains:

$$\begin{cases} \varepsilon_{\theta\theta} = u' - \theta_0 w + \frac{1}{2} V^2 + \dfrac{\rho - R}{R\theta_0} V' \\ \varepsilon_{\varphi\varphi} = u\theta_0 \cot\theta_0\eta - \theta_0 w + \dfrac{\rho - R}{R} V \cot\theta_0\eta \end{cases}$$

* Linear constitutive law:

$$\begin{cases} \sigma_{\theta\theta} = \dfrac{E}{1-\nu^2}\left(\varepsilon_{\theta\theta} + \nu\, \varepsilon_{\varphi\varphi}\right) \\ \sigma_{\varphi\varphi} = \dfrac{E}{1-\nu^2}\left(\varepsilon_{\varphi\varphi} + \nu\, \varepsilon_{\theta\theta}\right) \end{cases}$$

* EULER EQUATIONS

$$\begin{cases} -S\left(\frac{V}{\eta}+\theta_0\right)+\frac{h^2}{3R^2}\left[\frac{1}{(1-\nu^2)\theta_0^2}\mathcal{L}\left(\frac{V}{\eta}\right)-\left(\frac{V}{\eta}+\theta_0\right)(S+2M)\right]=0 \\ \mathcal{L}(S)+\frac{h^2}{3R^2}\mathcal{L}(M)+\frac{V}{\eta}\left(\frac{V}{2\eta}+\theta_0\right)=0 \end{cases}$$

$\rightarrow \mathcal{L} = \frac{1}{\eta^3}\frac{d}{d\eta}\left(\eta^3\frac{d}{d\eta}\right)$

$\rightarrow S, M$: membrane and bending parts of $\sigma_{\theta\theta}$

* BOUNDARY CONDITIONS

$$\begin{cases} S'(0)\left(1+\frac{h^2}{3R^2}\right)+\frac{2h^2}{3R^2}M'(0)=0, \\ \left(\frac{V}{\eta}\right)'(0)=0, \\ S(1)\left(1+\frac{h^2}{3R^2}\right)+\frac{2h^2}{3R^2}M(1)=\frac{P_0}{hE}, \\ \frac{1}{\theta_0}V'(1)+\frac{\nu}{\theta_0}V(1)=-(1-\nu^2)\frac{1-h^2/3R^2}{1-\frac{4h^2}{3R^2}}\frac{P_0}{hE}. \end{cases}$$

THE GENERIC CASE

$$\frac{h}{R\theta_0} \neq 0 , \quad \theta_0 \neq 0$$

This is the general case

* thin shell
* ≈ shallow shell

$$\begin{cases} f(\eta) = \dfrac{V(\theta_0 \eta)}{\eta} \\[2ex] g(\eta) = 2\left[S + \dfrac{h^2}{3R^2}(S + 2M)\right] + \dfrac{P_0}{hE} \end{cases}$$

EQUILIBRIUM EQUATIONS

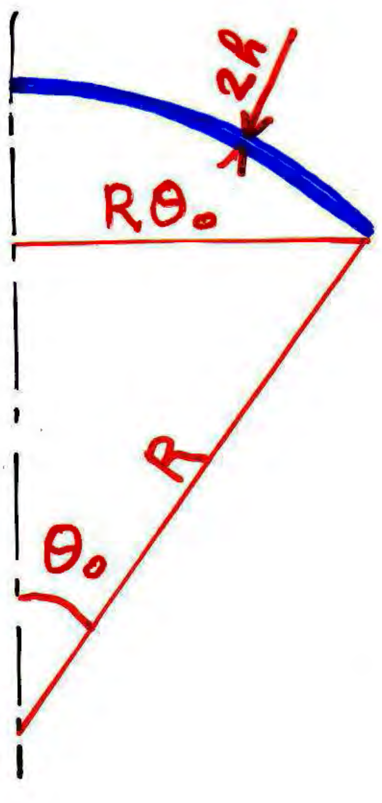
$\Longrightarrow$ A multiparameter bifurcation problem.

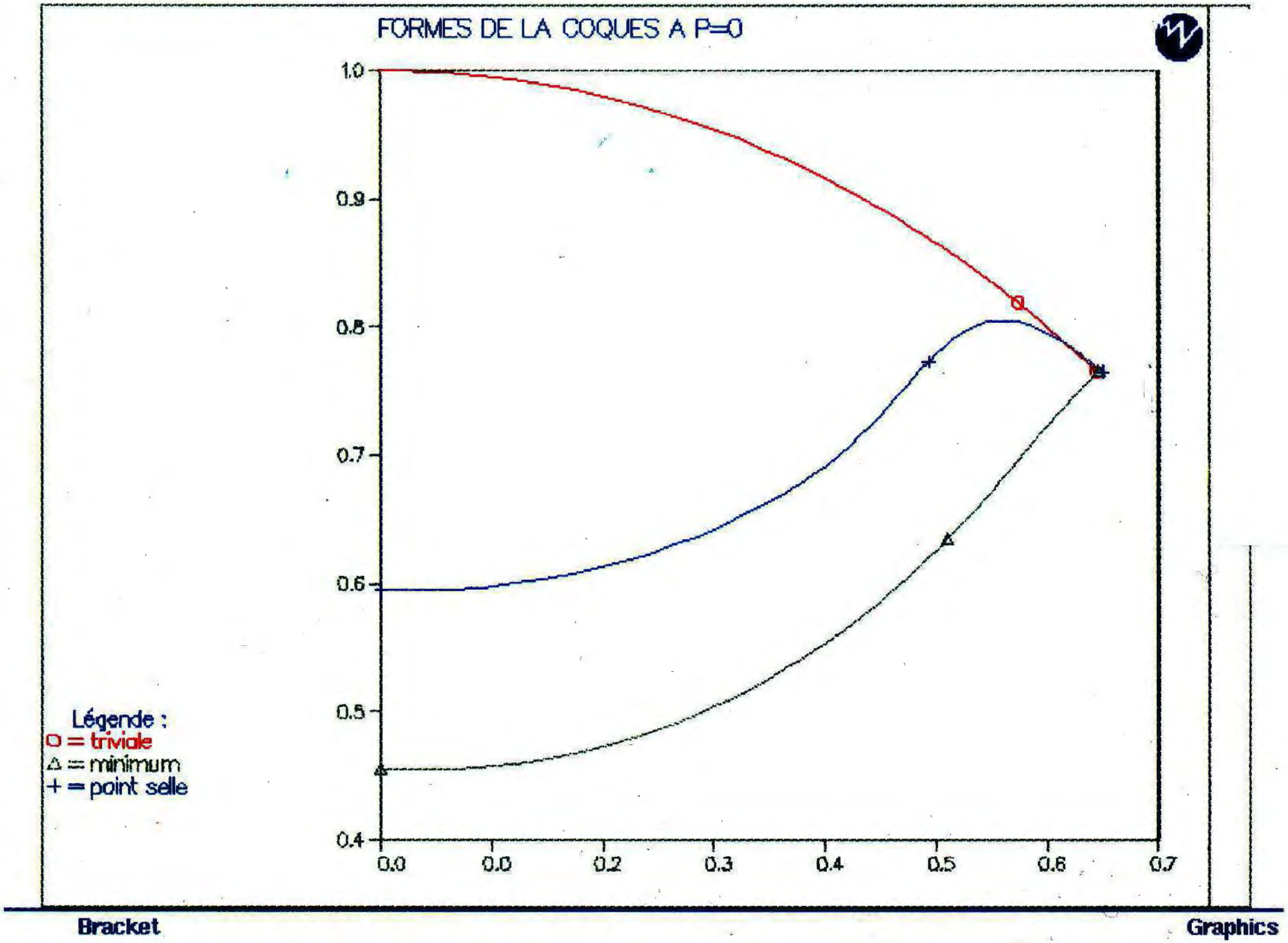
$$(\mathcal{P}): \boxed{\begin{array}{l} \left.\begin{array}{l} \mathcal{L}g = -f(f+2\lambda_3) \\ \mathcal{L}f = \lambda_1^2 (f+\lambda_3)(g+\lambda_2) \end{array}\right\} \eta \in]0,1[\\ f'(0) = g'(0) = g(1) = 0 \\ f'(1) + (1+\nu) f(1) = -(1-\nu^2)\lambda_2 \lambda_3 \end{array}}$$

$\rightarrow \lambda_1^2 = \dfrac{3R^2\theta_0^2}{2h^2}(1-\nu^2)$

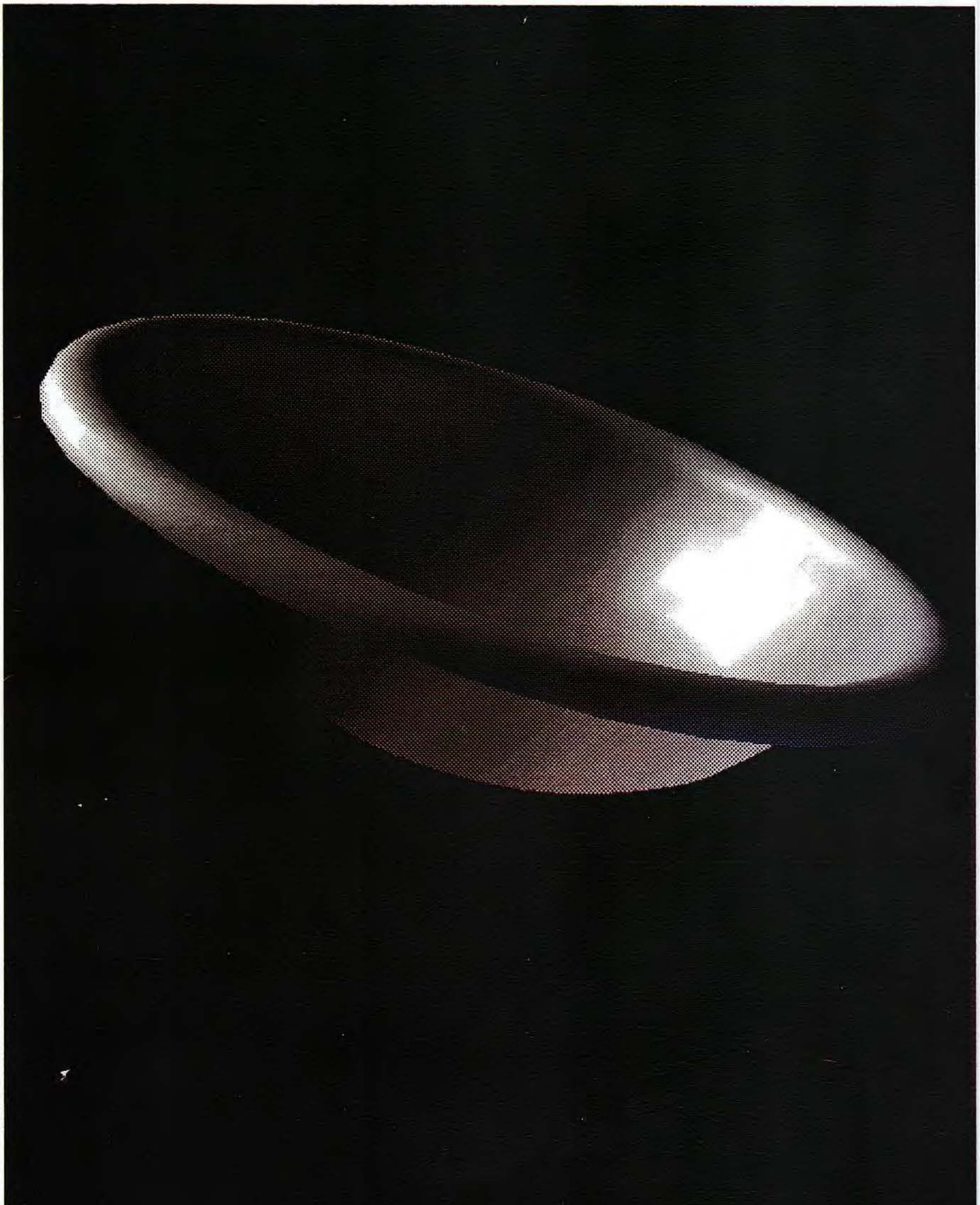
$\rightarrow \lambda_2 = \dfrac{P_0}{hE}$

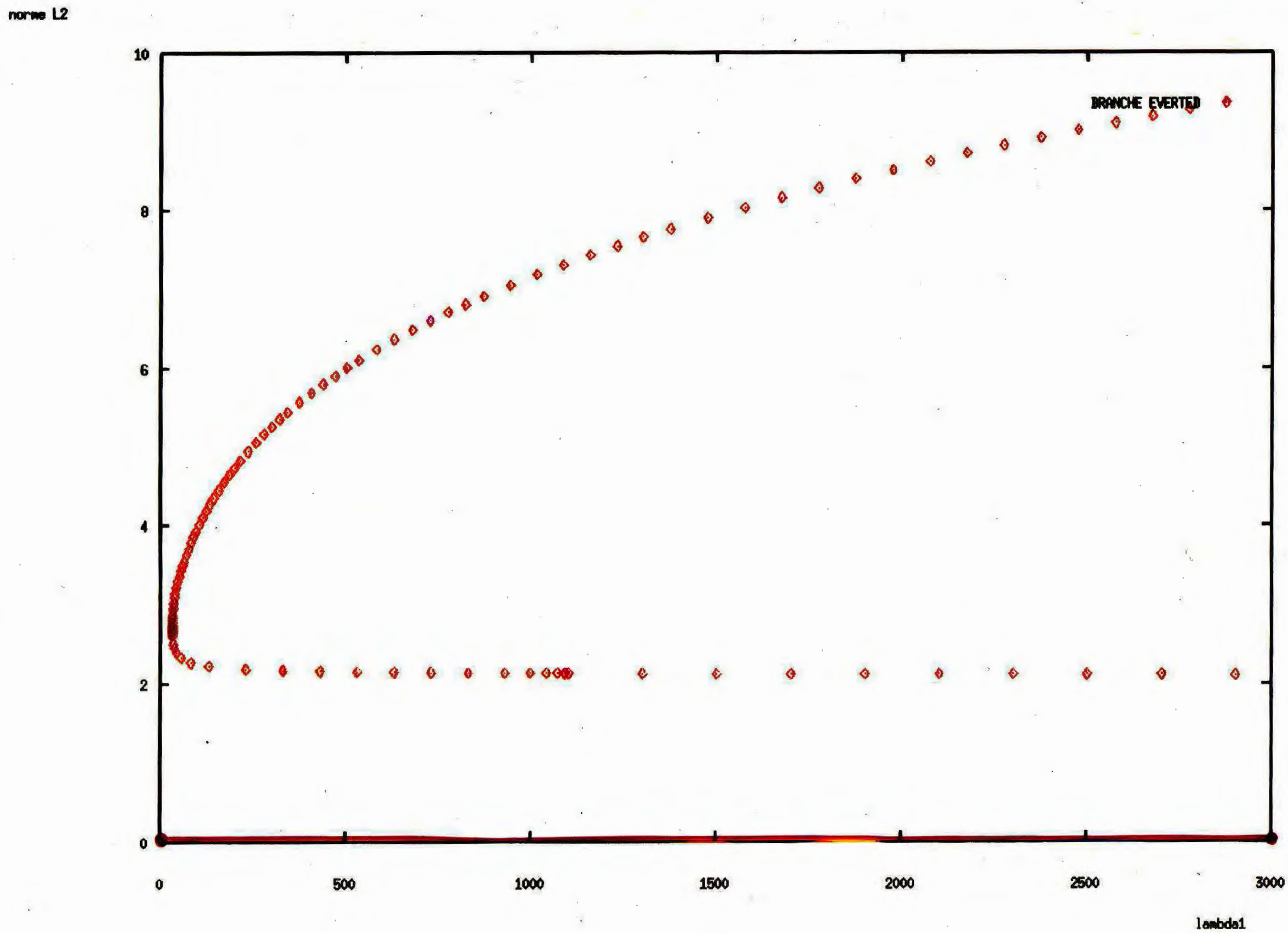
$\lambda_3 = \theta_0$





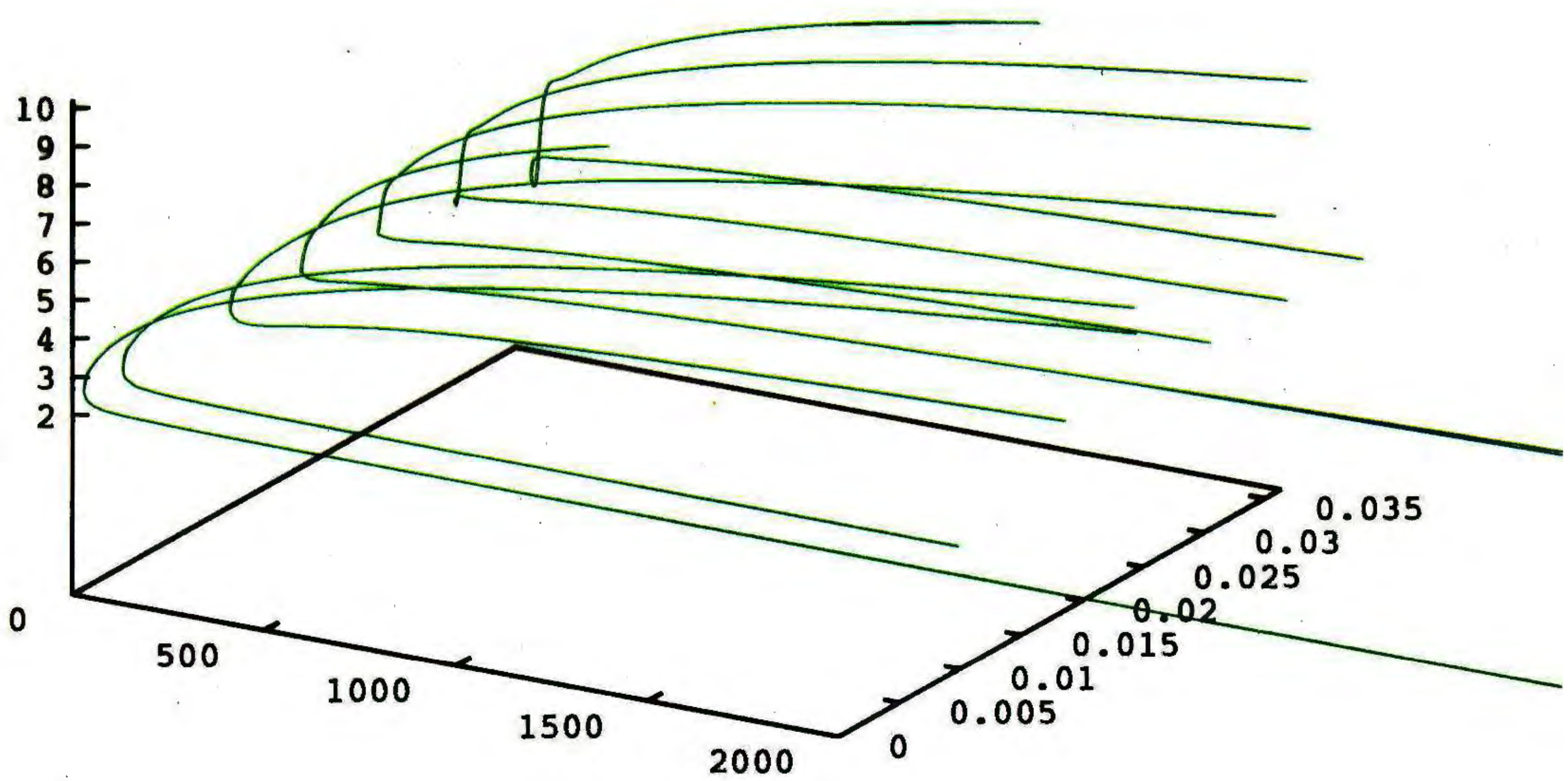
FORMES DE LA COQUES A P=0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.0
0.0
0.2
0.3
0.4
0.5
0.6
0.7
Légende :
O = triviale
Δ = minimum
+ = point selle
Bracket
Graphics

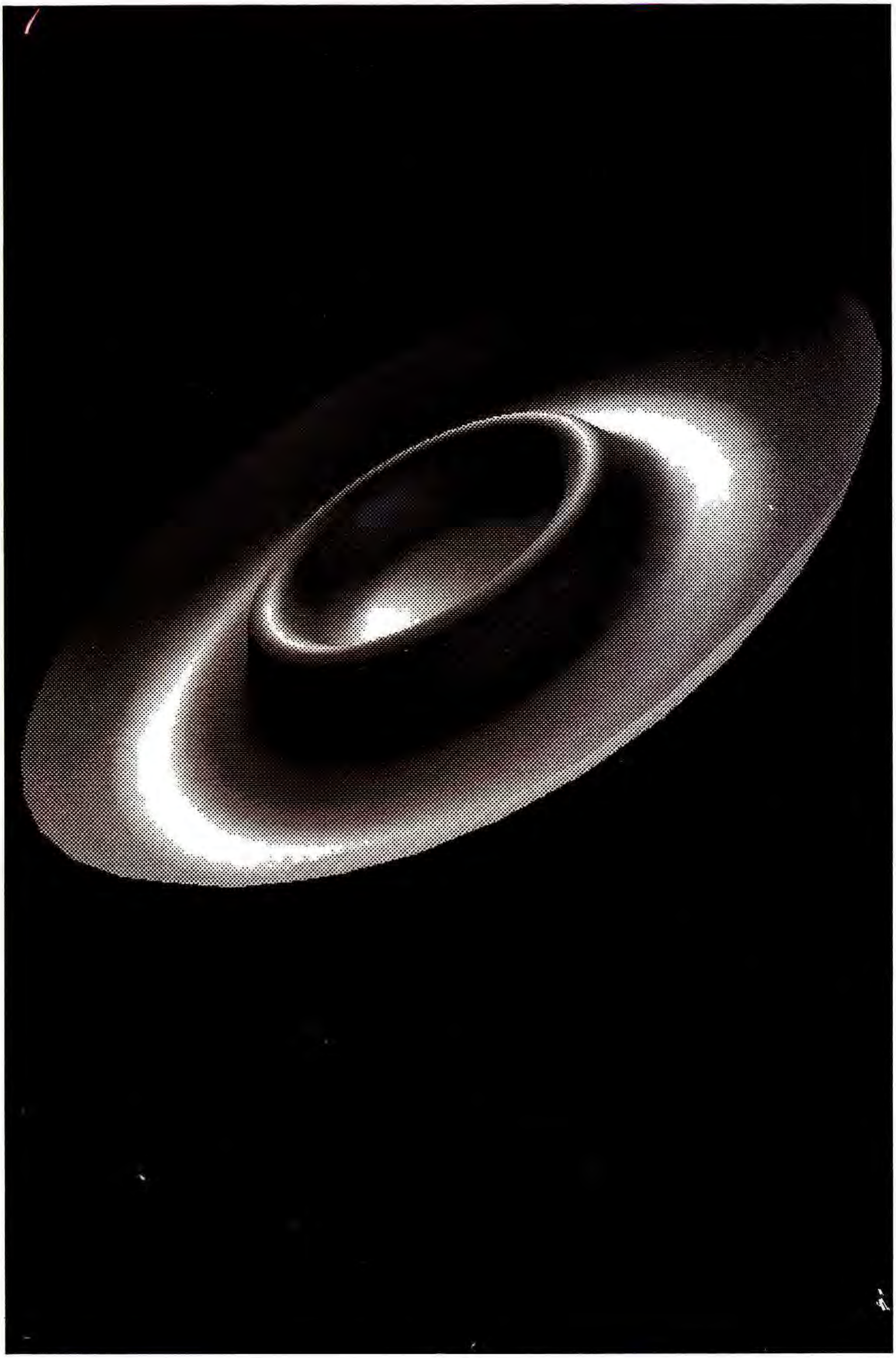




norme L2
BRANCHE EVERTED
10
8
6
4
2
0
0
500
1000
1500
2000
2500
3000
lambda1

EVOLUTION DE LA BRANCHE EVERTED AVEC LE CHARGEMENT





THE SHELL TENDS TO A PLATE

$$\frac{h}{R\theta_0} \neq 0 , \quad \theta_0 \longrightarrow 0$$

$$\left(0 < \lambda_1 < +\infty , \quad \lambda_3 \longrightarrow 0 \right)$$

$$\begin{cases} S \dfrac{V}{\eta} - \dfrac{1}{1-\nu^2} \dfrac{h^2}{3R^2\theta_0^2} \mathcal{L}\left(\dfrac{V}{\eta}\right) = 0 \\ \\ \mathcal{L}(S) + \dfrac{V^2}{2\eta^2} = 0 \end{cases} \qquad \eta \in]0,1[$$

$$\begin{cases} V(0) = 0 \\ S'(0) = 0 \\ V'(1) + \nu V(1) = 0 \\ S(1) = \dfrac{P_0}{2hE} \end{cases}$$

$\longrightarrow$ Von Kármán Plate!

THE SHELL TENDS TO A MEMBRANE

$$\lambda_1 \longrightarrow +\infty \,, \quad \lambda_3 > 0$$

$$\boxed{\begin{array}{l} \left.\begin{array}{l} \mathcal{L}(g) = -f(f+2\lambda_3) \\ (f+\lambda_3)(g+\lambda_2) = 0 \end{array}\right\} \eta \in]0,1[\\ g'(0) = g(1) = 0 \end{array}}$$

$* \ \lambda_2 = 0$: 1) $f \equiv 0 \ ; \quad g \equiv 0$

2) $f \equiv -2\lambda_3 \ ; \quad g \equiv 0$

3) $f \equiv -\lambda_3 \ ; \quad g = \frac{\lambda_3^2}{8}(\eta^2 - 1)$

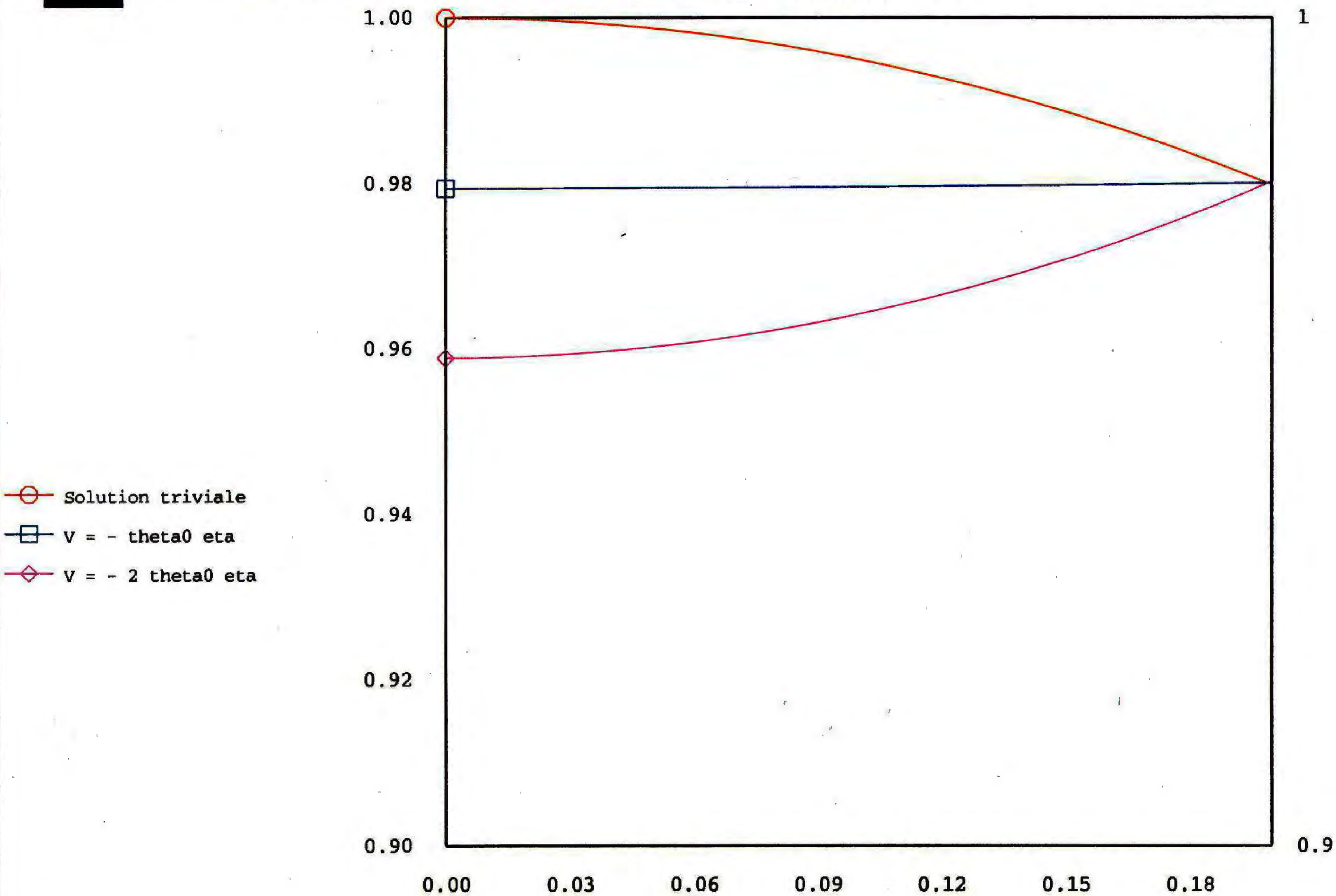
$* \ \lambda_2 \neq 0$: $f \equiv -\lambda_3 \ ; \quad g = \frac{\lambda_3^2}{8}(\eta^2 - 1)$

$\Longrightarrow$ "SMOOTH" SOLUTIONS

EDF

Département Mécanique et Modèles Numériques

SOLUTIONS DES EQUATIONS DE LA MEMBRANE EN L'ABSENCE DE FORCE



agraf

FUNCTIONAL FRAMEWORK AND CONSEQUENCES

$$\left.\begin{aligned} \mathcal{L} g &= -f_1 \cdot f_2 \\ g(0) &= g(1) = 0 \end{aligned}\right\} \Longrightarrow \boxed{g = \mathcal{G}_0(f_1, f_2)}$$

$f \in \mathbb{K} \equiv \left\{ \begin{array}{l} \text{square integrable fcts over }]0,1[\text{ with the} \\ \text{weight } \eta^3 \end{array} \right\}$

$g \in \mathbb{K}_0^1 \equiv \left\{ \begin{array}{l} \text{fcts of } \mathbb{K}, \text{ first derivatives in } \mathbb{K}, \\ \text{equal to zero at } \eta = 1 \end{array} \right\}$

$$\boxed{\begin{aligned} \Gamma(f; \lambda_2, \lambda_3) = \frac{1}{4} \int_0^1 \left[\frac{d\,\mathcal{G}_0(f, f + 2\lambda_3)}{d\eta} \right]^2 \eta^3 \, d\eta \\ + \frac{\lambda_2}{2} \int_0^1 (f + \lambda_3)^2 \, \eta^3 \, d\eta \end{aligned}}$$

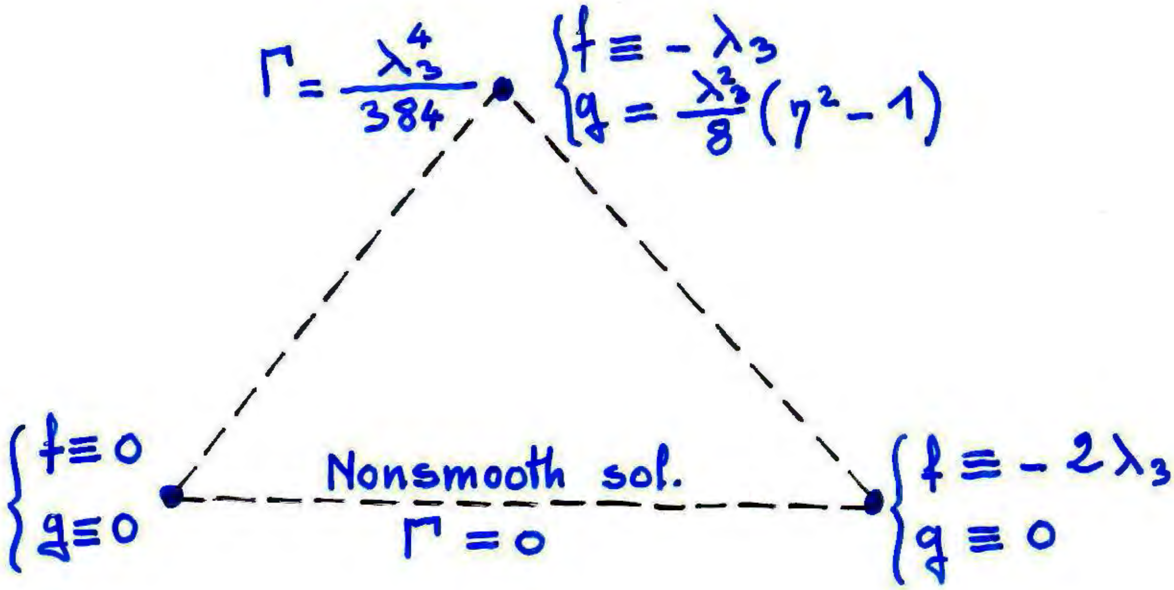
* $\{f, g\}$ is an equilibrium point

$\Updownarrow$

* f is a critical point of Γ

$< \Gamma'(f), \varphi > = 0, \ \forall \varphi \in \mathbb{K}$

$$\boxed{\lambda_2 = 0}$$



In addition to smooth solutions, the unloaded problem possesses infinitely many other solutions of two kinds

i) all the pairs (f, g), $g \equiv 0$, f satisfying $f(f + 2\lambda_3) = 0$ over $]0,1[$.

ii) all the pairs (f, g) s.t.

* if $g = 0$ then $f = 0$ or $f = -2\lambda_3$
* if $f = -\lambda_3$ then $\mathcal{L} g = -\lambda_3^2 \chi_E$

where: $\chi_E(\eta) = \begin{cases} 1 & \text{if } \eta \in E \\ 0 & \text{otherwise} \end{cases}$

$E = \{\eta \in]0,1[\,/\, f(\eta) = -\lambda_3\}$, measurable

PROPERTIES

OF THE CRITICAL POINTS

i) Classical solutions $\{f \equiv 0\,;\, g \equiv 0\}$ and $\{f = -2\lambda_3\,;\, g \equiv 0\}$ are MINIMA of Γ.

ii) Classical solution $\left\{f = -\lambda_3\,;\, g = \frac{\lambda_3^2}{8}(\eta^2 - 1)\right\}$ is a MAXIMUM of Γ.

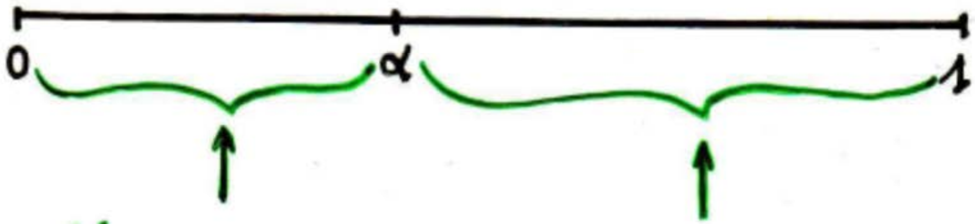
iii) Nonsmooth solutions involving only $f \equiv 0$ and $f = -2\lambda_3$, on any partition of $[0,1]$, are MINIMA of Γ.

iv) Any nonsmooth solution involving $f = -\lambda_3$ on any nonzero measure part of $[0,1]$ is a SADDLE POINT of Γ.

PROOF:

$$< \Gamma''(f).\psi, \psi > = \left\langle \mathcal{G}_0[f+\lambda_3, f+\lambda_3] - \mathcal{G}_0(\lambda_3,\lambda_3), \mathcal{G}_0(\psi,\psi)\right\rangle_1 + 2\left\langle \mathcal{G}_0(\psi, f+\lambda_3), \mathcal{G}_0(\psi, f+\lambda_3)\right\rangle_1$$

$\rightarrow$ i), iii) and ii): straightforward.



$$\begin{cases} f = 0 \\ g = 0 \end{cases} \qquad \begin{cases} f = -\lambda_3 \\ g = \dfrac{\lambda_3^2}{8}\left(\eta^2 + \dfrac{\alpha^2}{\eta^2} - \alpha^2 - 1\right) \end{cases}$$

$$\langle \Gamma''(f)\cdot\varphi, \varphi\rangle = \int_0^\alpha \left[\frac{d}{d\eta}\, g_0(\varphi, \lambda_3)\right]^2 \eta^3\, d\eta$$

$$+ \frac{\lambda_3^2}{8}\int_\alpha^1 \left(\eta^2 + \frac{\alpha^2}{\eta^2} - \alpha^2 - 1\right)\varphi^2 \eta^3\, d\eta .$$

$$\Downarrow$$

SADDLE POINT.