SHALLOW NONLINEAR MEMBRANES

$$f^\varepsilon \in L^2(\omega)$$
$$\varphi_1^\varepsilon \in H^{3/2}(\partial\omega)$$
$$\varphi_2^\varepsilon \in H^{1/2}(\partial\omega)$$

$$\boxed{\begin{array}{l} \left\{\begin{aligned} \frac{2E}{3(1-\nu^2)}\,\varepsilon^3 \Delta^2 u_3^\varepsilon &= 2\varepsilon\left[\varphi^\varepsilon, u_3^\varepsilon + \theta^\varepsilon\right] + f^\varepsilon \\ \Delta^2 \varphi^\varepsilon &= -\frac{E}{2}\left[u_3^\varepsilon, u_3^\varepsilon + 2\theta^\varepsilon\right] \end{aligned}\right. \\ \text{in } \omega; \\ \left\{\begin{aligned} u_3^\varepsilon &= \partial_\nu u_3^\varepsilon = 0 \\ \varphi^\varepsilon &= \varphi_1^\varepsilon, \quad \partial_\nu \varphi^\varepsilon = \varphi_2^\varepsilon, \end{aligned}\right. \\ \text{on } \partial\omega. \end{array}}$$

$$[f, g] := \partial_{11} f \cdot \partial_{22} g + \partial_{22} f \cdot \partial_{11} g - 2\,\partial_{12} f \cdot \partial_{12} g.$$

φ : Airy function associated with u_1 and u_2

UNLOADED MEMBRANES

2 STEPS:

* $f^{\varepsilon} = 0$

$\varphi_1^{\varepsilon} = \varphi_2^{\varepsilon} = 0$

** $\varepsilon = 0$

$\hookrightarrow u^{\varepsilon}, \varphi^{\varepsilon}, S^{\varepsilon}, \theta^{\varepsilon} \longrightarrow u, \varphi, S, \theta$

$$(\mathcal{P}) \quad \boxed{\begin{array}{ll} \left.\begin{array}{l} [\varphi, u+\theta] = 0 \\ \Delta^2 \varphi = -\frac{E}{2}[u, u+2\theta] \end{array}\right\} & \text{in } \omega \\ \left.\begin{array}{l} u = 0 \\ \varphi = \partial_\nu \varphi = 0 \end{array}\right\} & \text{on } \partial\omega \end{array}}$$

3 ASSUMPTIONS:

* θ is as smooth as necessary,

** S is convex,

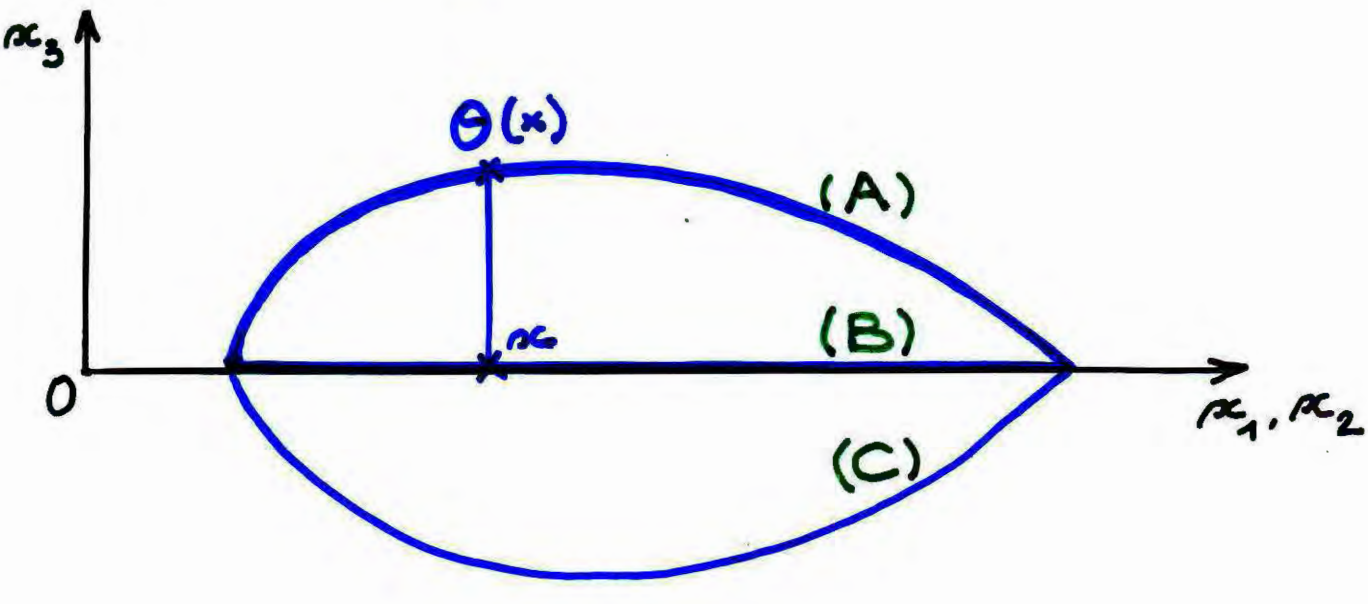
*** $\theta = 0$ on $\partial\omega$.

Problem $(\mathcal{P})$ possesses 3 elementary solutions :

$$(A) \begin{cases} u \equiv 0 \\ \varphi \equiv 0 \end{cases} \quad \text{in } \overline{\omega}$$

$$(B) \begin{cases} u = -\theta \quad \text{in } \overline{\omega} \\ \Delta^2 \varphi = \frac{E}{2}[\theta, \theta] \quad \text{in } \omega \\ \varphi = \partial_\nu \varphi = 0 \quad \text{on } \partial\omega \end{cases}$$

$$(C) \begin{cases} u = -2\theta \\ \varphi \equiv 0 \end{cases} \quad \text{in } \overline{\omega}$$



Let : $\mathbb{K}(\omega) = H_0^1(\omega) \cap H^2(\omega)$

and : $$\begin{cases} \Delta^2 \Psi = -\frac{E}{2}[v,w] \text{ in } \omega \\ \Psi = \partial_\nu \Psi = 0 \text{ on } \partial\omega \\ \forall\, v, w \text{ given in } \mathbb{K} \end{cases}$$

$$\left| \begin{array}{l} \Psi \in \mathbb{H}_0^2(\omega) \\ \displaystyle\int_\omega \Delta\Psi \, \Delta z \, d\omega = -\frac{E}{2}\int_\omega [v,w]\, z \, d\omega \\ \forall z \in \mathbb{H}_0^2(\omega),\ \forall (v,w) \in \mathbb{K} \times \mathbb{K} \end{array} \right.$$

$$\mathcal{G} : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{H}_0^2(\omega)$$
$$(v,w) \longmapsto \mathcal{G}(v,w) = \Psi$$

* $\mathcal{G}(\cdot,\cdot)$ bilinear, symmetric

** $\mathcal{G}(\cdot, v)$ linear, compact, self-adjoint $\forall v \in \mathbb{K}$

*** $w \longmapsto \mathcal{G}(w,w)$ is compact in $\mathbb{K}$.

Let a partition P

$$P : \quad \omega = \bigcup_{i=1}^{N} \omega_i \ , \ \omega_i \cap \omega_j = \emptyset$$

and :

$$\mathbb{K}_1(\omega) = \left\{ v \in \mathbb{H}_0^1(\omega) \Big/ v\big|_{\omega_i} \in \mathbb{H}^2(\omega_i) \right\}$$

$$\mathbb{K}_2(\omega) = \left\{ v \in \mathbb{H}_0^2(\omega) \Big/ v\big|_{\omega_i} \in \mathbb{H}_0^2(\omega_i) \right\}$$

THEN: $\mathcal{G}_P : \mathbb{K}_1 \times \mathbb{K}_1 \longrightarrow \mathbb{K}_2$ defined by

$$\left| \begin{array}{l} \Psi \in \mathbb{K}_2 \ \Big/ \ \Psi\big|_{\omega_i} \underset{def}{:=} \Psi_i \\ \Delta^2 \Psi_i = -\frac{E}{2}\left[v\big|_{\omega_i} , w\big|_{\omega_i} \right] \\ \Psi_i = \partial_\nu \Psi_i = 0 \quad \text{on } \partial\omega_i \\ \forall \ v \text{ and } w \text{ given in } \mathbb{K}_1 \end{array} \right.$$

$$\longrightarrow \mathcal{G}_P(v, w) = \Psi$$

Let then :

$$\left| \begin{array}{l} \Gamma_P(v) = \displaystyle\int_\omega \left(\Delta \, \mathcal{G}_P(v, v+2\theta) \right)^2 d\omega \\ \forall v \in \mathbb{K}_1(\omega) \end{array} \right.$$

THEOREM:

Let $u \in \mathbb{K}_1$ be such that $\Gamma_P'(u) \cdot v = 0, \ \forall v \in \mathbb{K}_1$. Then (u, φ), $\varphi = \mathcal{G}_P(u, u+2\theta)$ is a solution to problem $(\mathcal{P})$.

PROOF:

Calculus of variations

+

LEMMA:

$\forall (f, g, h)$, $f \in \mathbb{K}_2$
$(g, h) \in \mathbb{K}_1 \times \mathbb{K}_1$:
$$\int_\omega f [g, h] \, d\omega = \int_\omega [f, g] \, h \, d\omega$$

$$\Gamma_P'(u) \cdot v = \int_\omega \Delta \mathcal{G}_P(u, u+2\theta) \, \Delta \mathcal{G}_P(v, u+\theta) \, d\omega$$

$$= \int_\omega \mathcal{G}_P(u, u+2\theta) [v, u+\theta] \, d\omega$$

$$= \int_\omega [\varphi, u+\theta] \, v \, d\omega. \quad \blacksquare$$

CHECK:

Elementary solutions A, B, and C are actually stationary points of $\Gamma_p(v)$ for $N = 1$.

COROLLARY:

If a partition of domain ω is such that there exists a solution to equation $[u, u + 2\theta] = 0$ in one of the subdomains, then we can build a solution to problem $(\mathcal{P})$ by combining this solution with elementary solutions A, B, C in the other subdomains.

$\Rightarrow$ Such a partition exists, and can be chosen in infinitely many ways!

THEOREM (P.L. Lions, T. Aubin)

Let $\Omega \subset \mathbb{R}^2$ be a regular, convex open set, ϕ and f regular given functions, f positive on Ω.

Then:

i) The problem $\begin{cases} \mathrm{Det}(\partial^2 w) = f & \text{in } \Omega \\ w = \phi & \text{on } \partial\Omega \end{cases}$

has a unique solution in $\mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$.

ii) If in addition $f \in \mathcal{C}^\infty(\bar{\Omega})$ and ϕ is the restriction to $\partial\Omega$ of a $\mathcal{C}^\infty(\bar{\Omega})$ function, then the solution is also in $\mathcal{C}^\infty(\bar{\Omega})$.

<u>Then:</u>

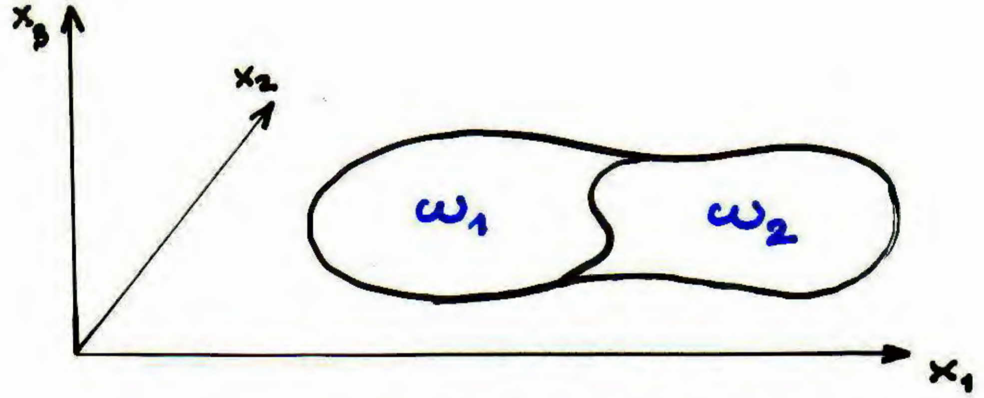
Let $v = u + \theta$, $[\theta, \theta] > 0$,

$$\Longrightarrow \left| \begin{array}{l} [v, v] = [\theta, \theta] \text{ in } \omega_2 \\ v = 0 \text{ on } \partial\omega_2 \end{array} \right.$$

has a unique $\mathcal{C}^\infty(\bar{\omega}_2)$ solution.

THEOREM :

Problem ($\mathcal{P}$) possesses infinitely many other solutions.



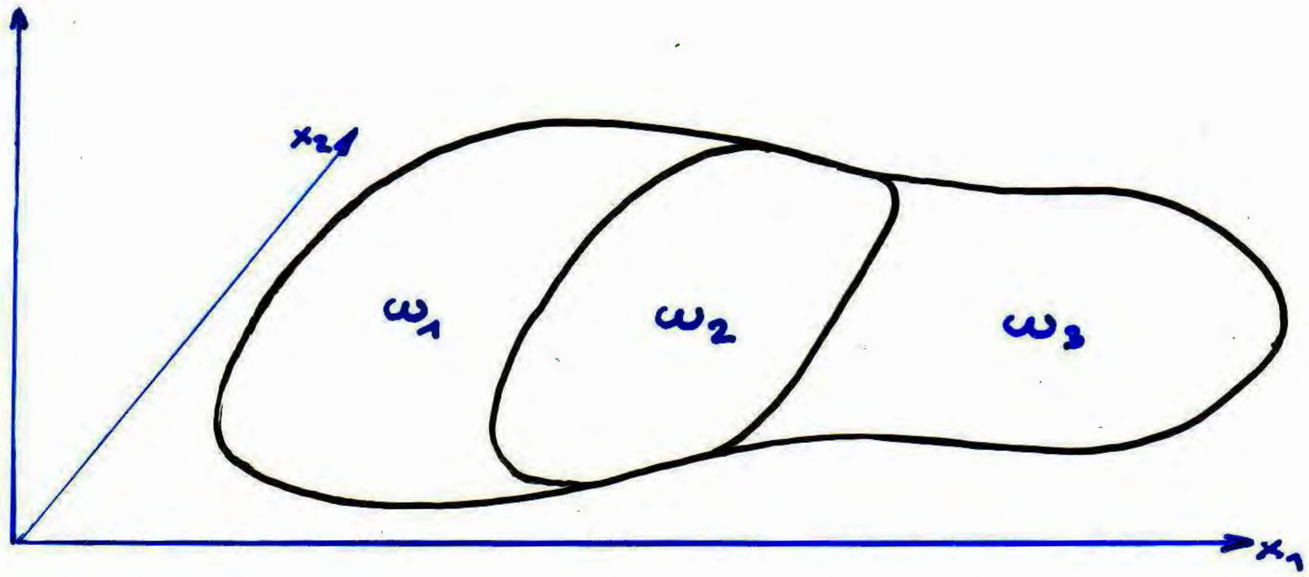
$$P_2 : \begin{cases} \omega = \omega_1 \cup \omega_2 \\ \omega_1 \cap \omega_2 = \phi \end{cases}$$

Let :

$$(\mathcal{P}_{P_2}) \quad \begin{cases} u = -\theta & \text{in } \overline{\omega_1} \\ \Delta^2 \varphi = \frac{E}{2}[\theta, \theta] & \text{in } \omega_1 \\ \varphi = \partial_\nu \varphi = 0 & \text{on } \partial\omega_1 \end{cases} \qquad \begin{cases} [u, u + 2\theta] = 0 & \text{in } \omega_2 \\ u = -\theta & \text{on } \partial\omega_2 \\ \varphi = 0 & \text{in } \overline{\omega_2} \end{cases}$$

Then :

A solution of problem ($\mathcal{P}_{P_2}$) is a stationary point of functional Γ_{P_2}.



$\mathbb{H}$:

$\omega = \omega_1 \cup \omega_2 \cup \omega_3$

$\omega_i \cap \omega_j = \phi$

ω_2 regular convex

$\gamma_i = \partial\omega_i \cap \partial\omega$

$\gamma_{ij} = \partial\omega_i \cap \partial\omega_j$

$\gamma_i \neq \phi \, , \quad i \in \{1, 2, 3\}$

$\gamma_{12} \neq \phi$

$\gamma_{23} \neq \phi$

$\gamma_{13} = \phi$

Let

$$(\mathcal{P}_{AB}) \quad (\text{resp. } \mathcal{P}_{BC})$$

$$\begin{cases} u = -\theta & \text{in } \overline{\omega_1} \\ \Delta^2 \varphi = \frac{E}{2}[\theta,\theta] & \text{in } \omega_1 \\ \varphi = \partial_\nu \varphi = 0 & \text{on } \partial\omega_1 \end{cases}$$

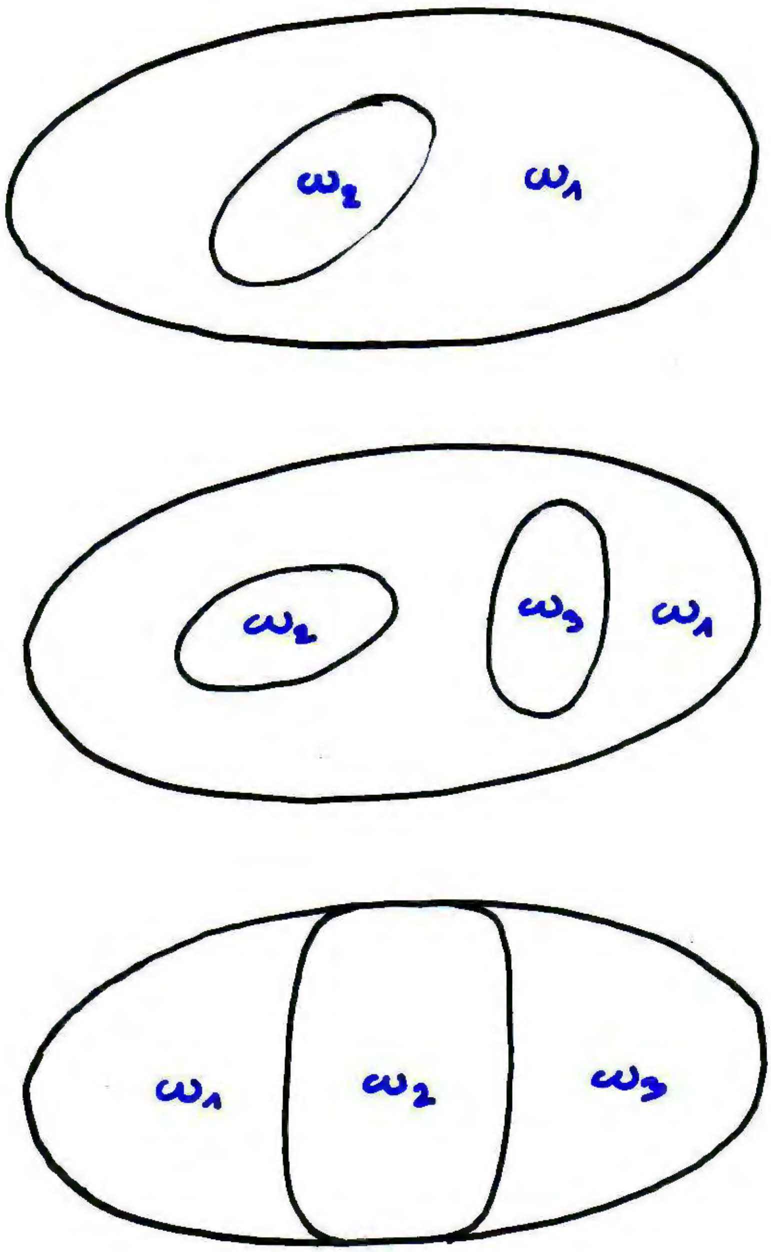
$$\begin{cases} [u, u + 2\theta] = 0 & \text{in } \omega_2 \\ u = -\theta & \text{on } \gamma_{12} \\ u = 0 \ (\text{resp. } -2\theta) & \text{on } \gamma_{23} \\ u = 0 & \text{on } \gamma_2 \\ \varphi = 0 & \text{in } \overline{\omega_2} \end{cases}$$

$$\left.\begin{cases} u = 0 \ (\text{resp. } -2\theta) \\ \varphi = 0 \end{cases}\right. \quad \text{in } \overline{\omega_3}$$

→ Problem $(\mathcal{P}_{AB})$ (resp. $\mathcal{P}_{BC}$) has a unique solution for any partition satisfying assumptions H.

→ There are many other solutions.

ABOUT THE SET OF SOLUTIONS



$\vdots$

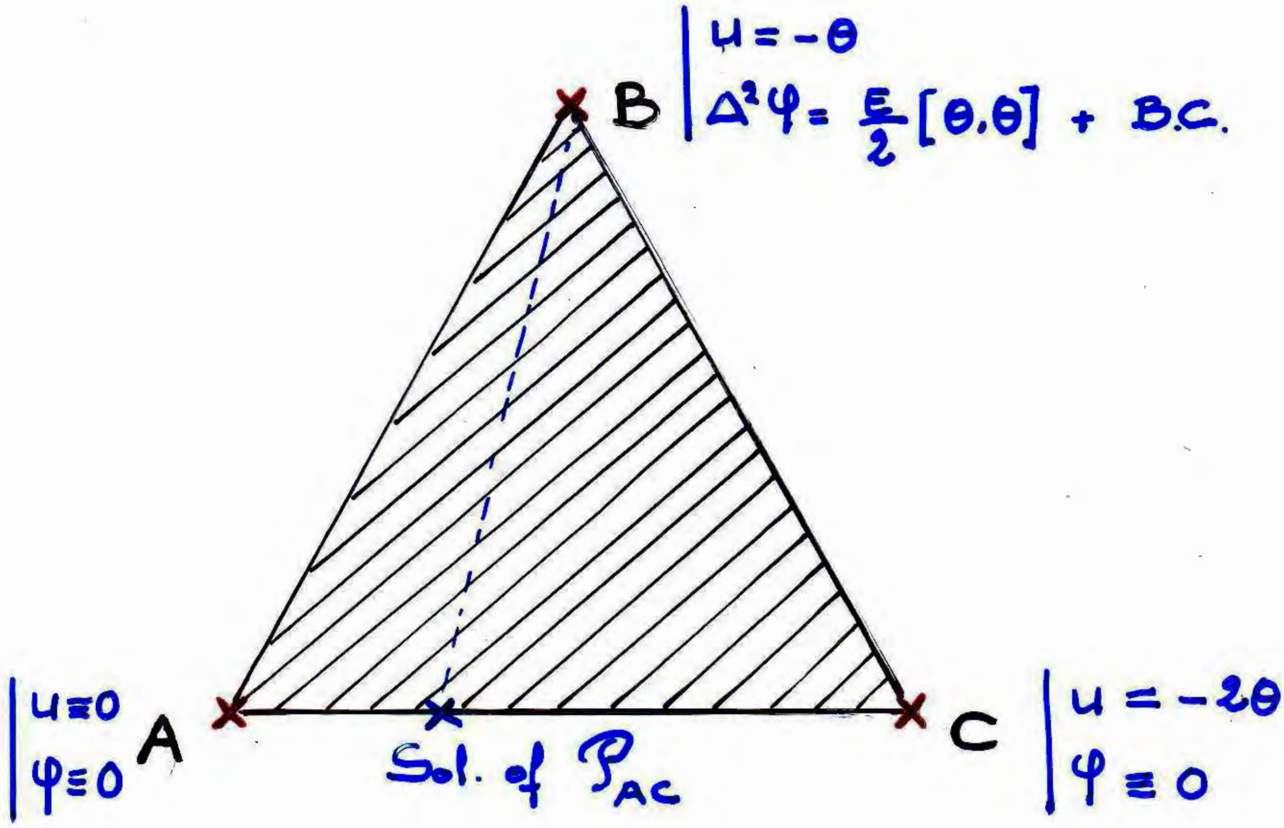
$$\omega = \bigcup_{i=1}^{N} \omega_i \quad \forall N < \infty$$

$$\omega_i \cap \omega_j = \emptyset$$

+

smoothness assumptions

$\rightarrow \ (u, \varphi) \in \mathbb{K}_1(\omega) \times \mathbb{K}_2(\omega)$



$\hookrightarrow$

- * What about critical point theory?
- * What about the topological structure of this triangle?
- * What about bifurcation theory?
- * Are there other solutions?

THEOREM:

i) Any solution built only with fonctions of $\mathbb{K}_1$ satisfying $[u, u+2\theta] = 0$ is a MINIMUM of Γ_p.

ii) Any solution involving $u = -\theta$ on any nonzero measure subdomain of ω is a SADDLE-POINT.

iii) Any solution locally everted with respect to a nonhorizontal plane is a SADDLE-POINT of Γ_p.

PROOF:

$$\langle \Gamma_p'(u), v \rangle = \int_\omega \Delta g_p(u, u+2\theta)\, \Delta g_p(v, u+\theta)\, d\omega$$

$$\langle \Gamma_p''(u).v, w \rangle = \int_\omega \Big\{ 2\Delta g_p(v, u+\theta)\, \Delta g_p(w, u+\theta) + \Delta g_p(u, u+2\theta)\, \Delta g_p(v, w) \Big\} d\omega$$

THEN

* i) Trivial.

PROOF (continued):

** $u = -\theta$ in ϖ

$$\langle \Gamma_p''(-\theta).v, v\rangle = -\int_\omega \Delta g_p(\theta,\theta)\,\Delta g_p(v,v)\,d\omega$$

$$= -\int_\omega \dot{g}_p(\theta,\theta)\,[v,v]\,d\omega$$

$\longrightarrow$ SADDLE - POINT

*** $u = -\theta$ in ω_1

$[u, u+2\theta] = 0$ in ω_2

$\omega_1 \cap \omega_2 = \emptyset$, $\omega_1 \cup \omega_2 = \omega$

$$\langle \Gamma_p''(u).v, v\rangle = -\int_{\omega_1} g_p(\theta,\theta)\,[v,v]\,d\omega$$

$$+\int_\omega \left(\Delta g_p(v, u+\theta)\right)^2 d\omega$$

.

.

.

.

QED

Let $\lambda = \frac{2}{3(1-\nu^2)} \xi^2$

$$(\mathcal{P}_\lambda) \left| \begin{array}{l} \lambda \ \Delta^2 u = [\varphi , u + \theta] \\ \quad \Delta^2 \varphi = - \frac{E}{2} [u, u + 2\theta] \\ + \text{ Homogeneous B. C.} \end{array} \right.$$

Let as previously

$$\varphi = - \frac{E}{2} \mathcal{G}(u, u + 2\theta)$$

Then :

$$\boxed{(\mathcal{P}_\lambda) \Longleftrightarrow \lambda u + \mathcal{G}\big(\mathcal{G}(u, u+2\theta), u+\theta\big) = 0}$$

↳ Non linear eigenvalue pb. for a functional operator.

Let then :

$$\left| \begin{array}{l} \Gamma_\lambda(v) = \frac{\lambda}{2} \int_\omega (\Delta v)^2 \, d\omega \\ \qquad + \frac{1}{4} \int_\omega \big(\Delta \mathcal{G}(v, v + 2\theta)\big)^2 \, d\omega \end{array} \right.$$

THEOREM:

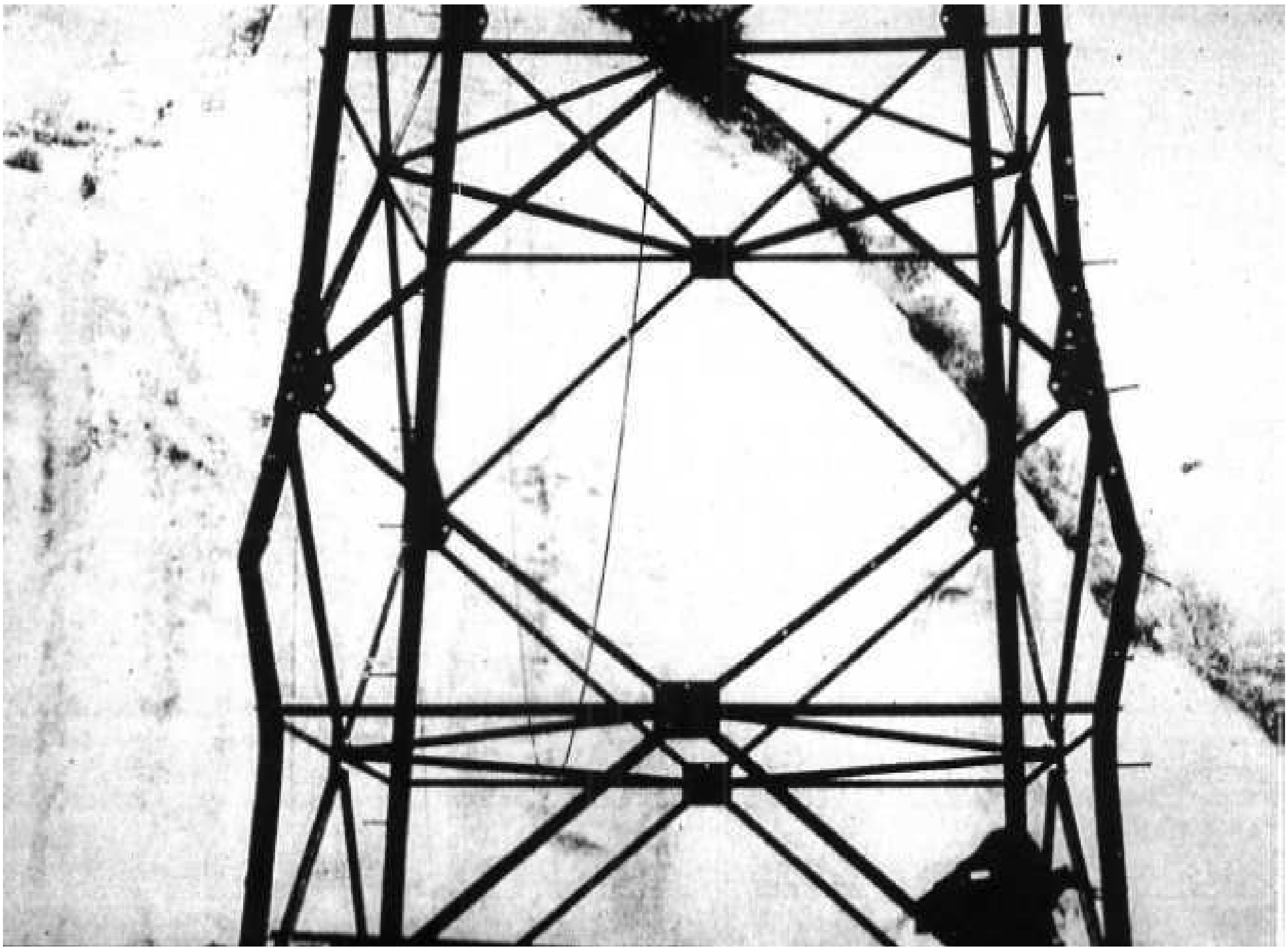
* $\lambda = 0$: Problem $(\mathcal{P}_\lambda)$ possesses infinitely many solutions.

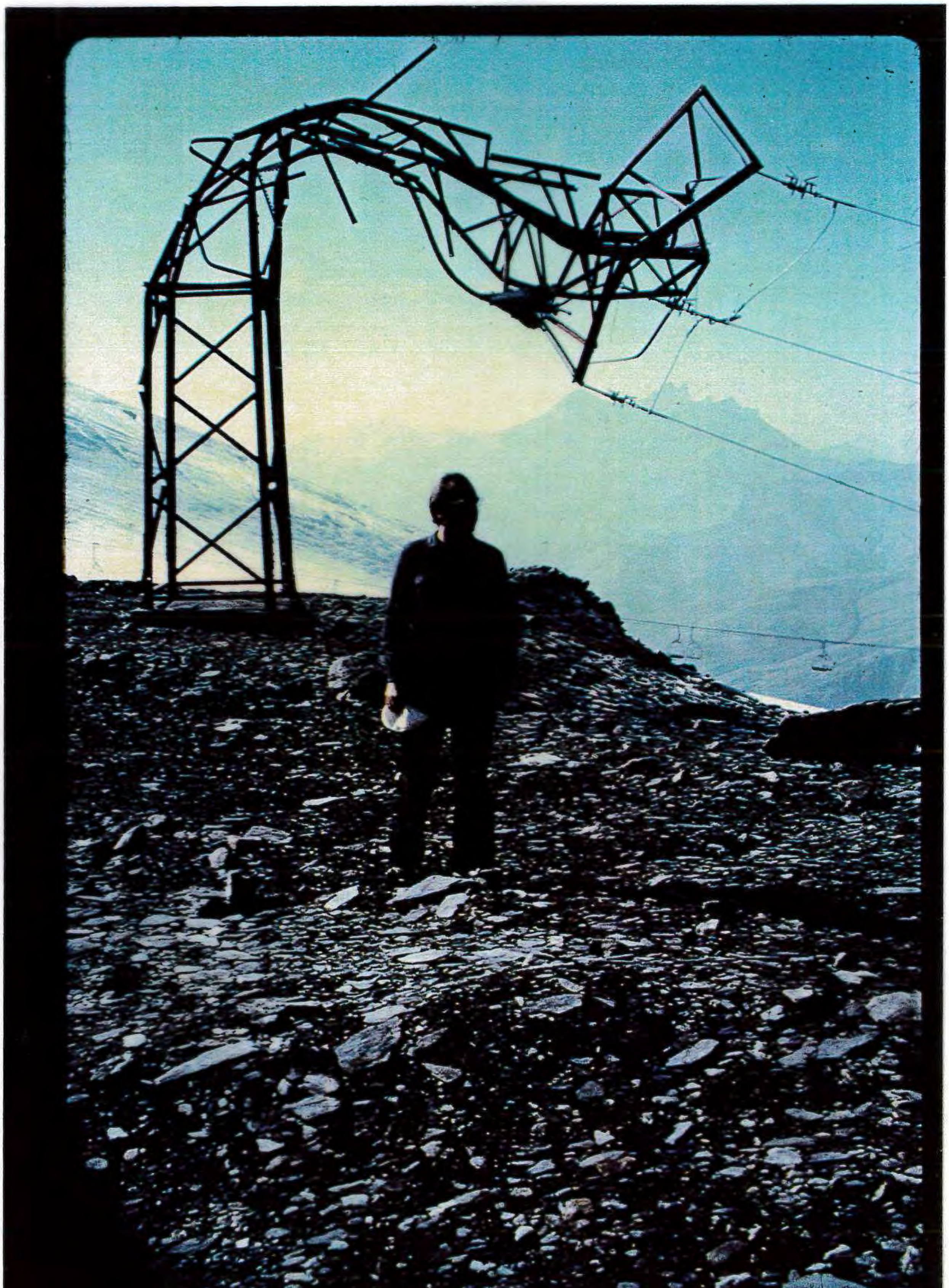
* $\lambda \neq 0$: The following statements are equivalent:
  - u is a stationary point of $\Gamma_\lambda(v)$
  - (u, φ) is a solution to problem $(\mathcal{P}_\lambda)$.

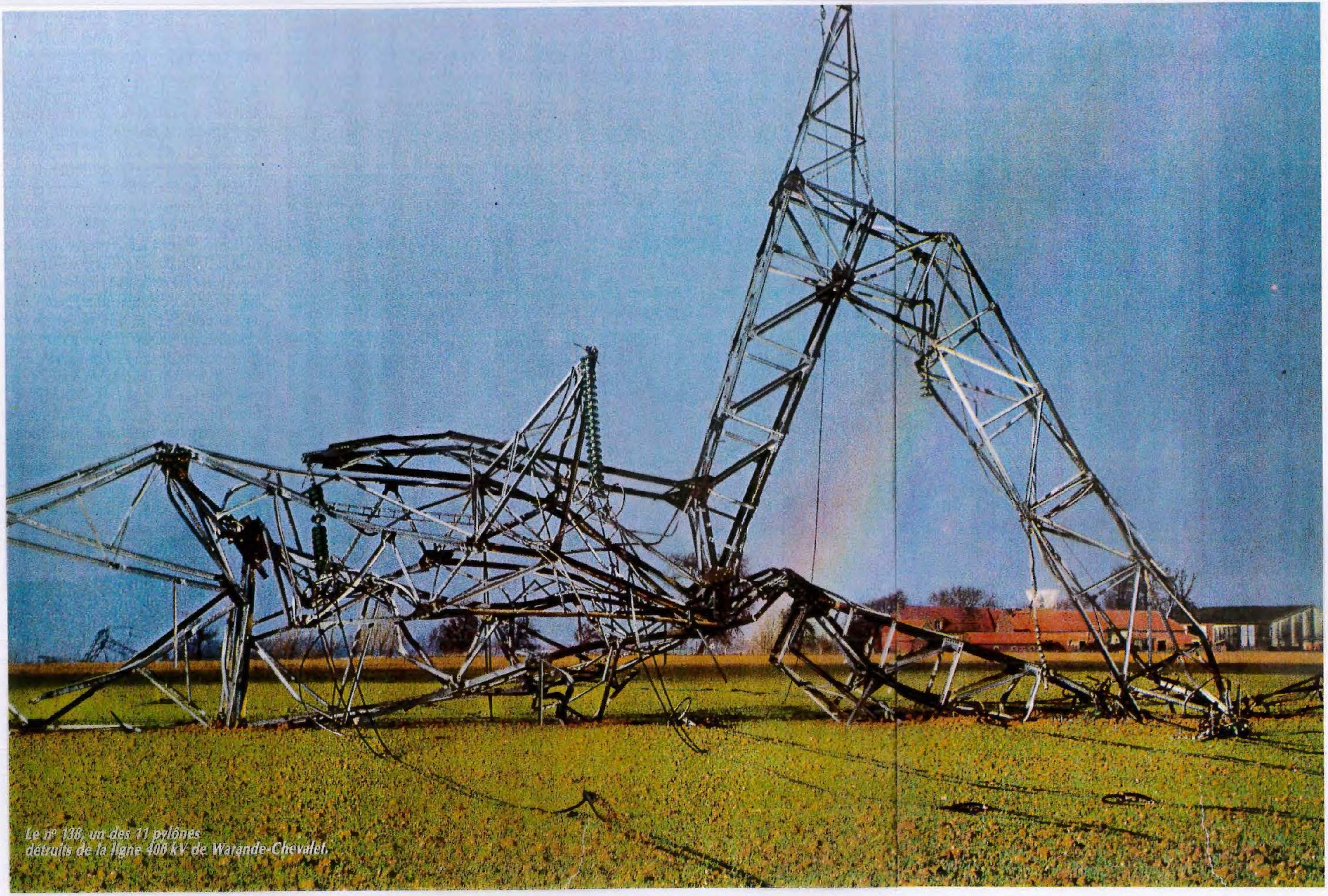
* $\exists\, \lambda_0 > 0$, s.t. $\forall\, \lambda > \lambda_0$ $\Gamma_\lambda(v)$ is strictly convex.

PLASTIC BUCKLING

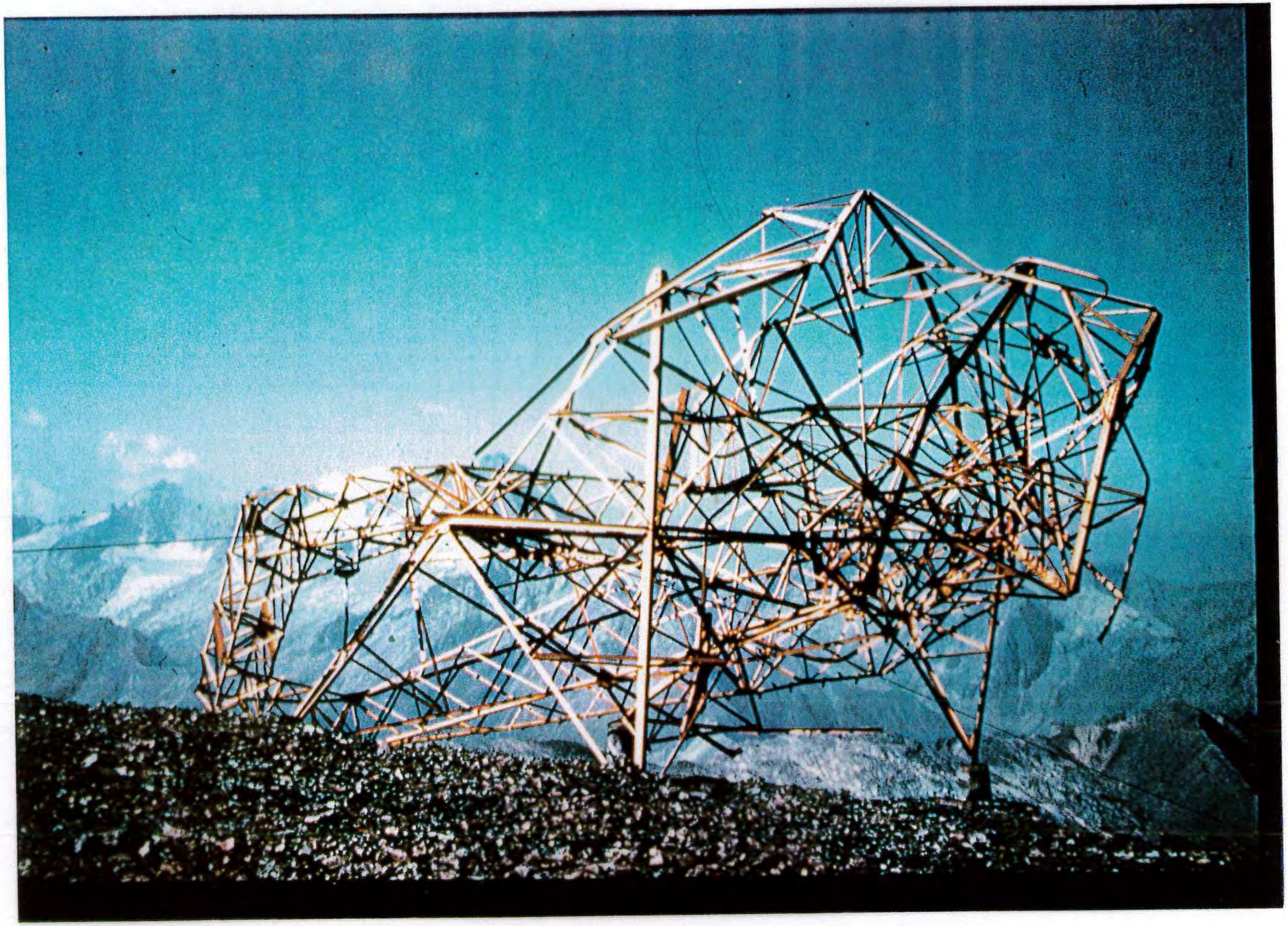
SOME OBSERVATIONS

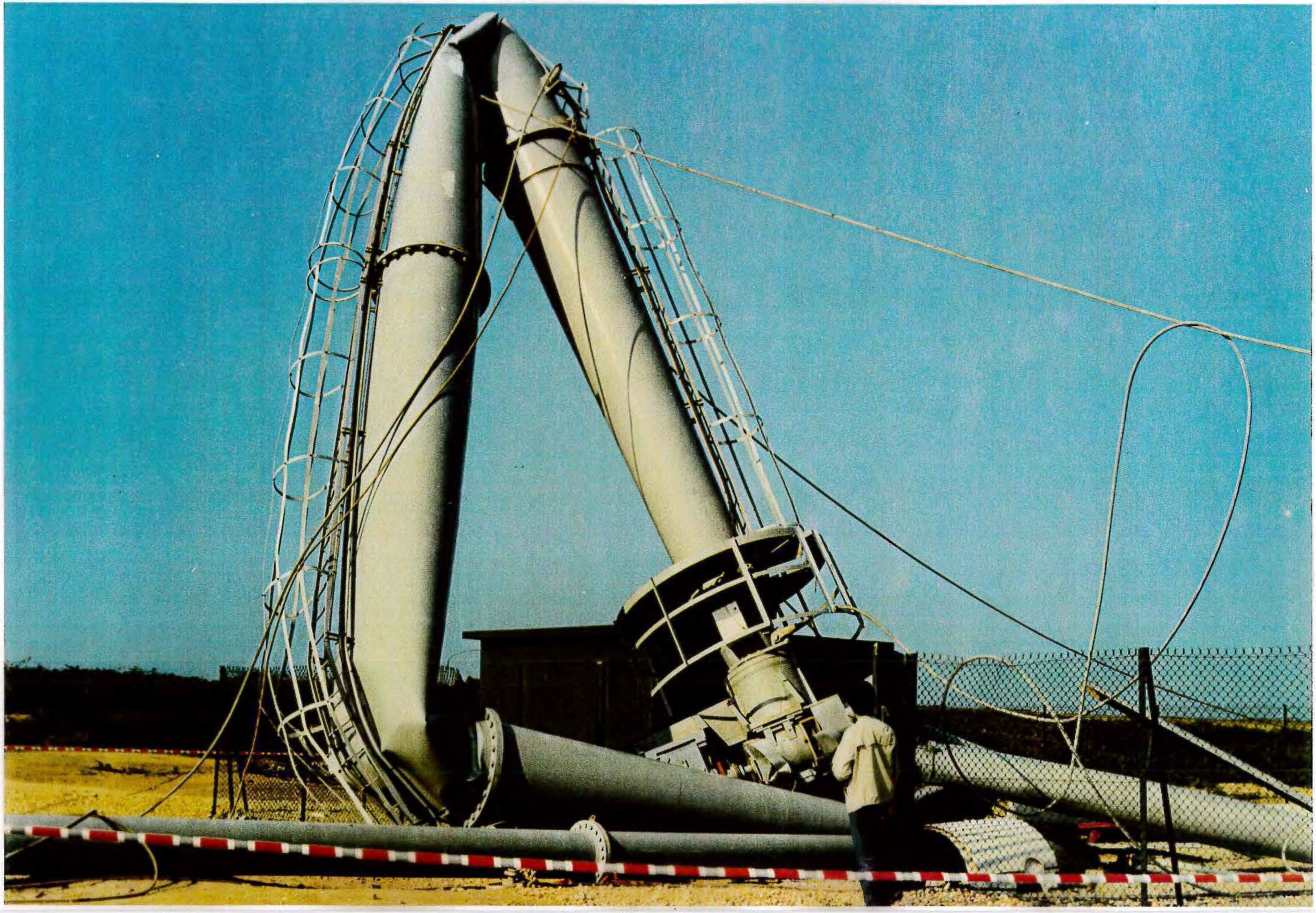


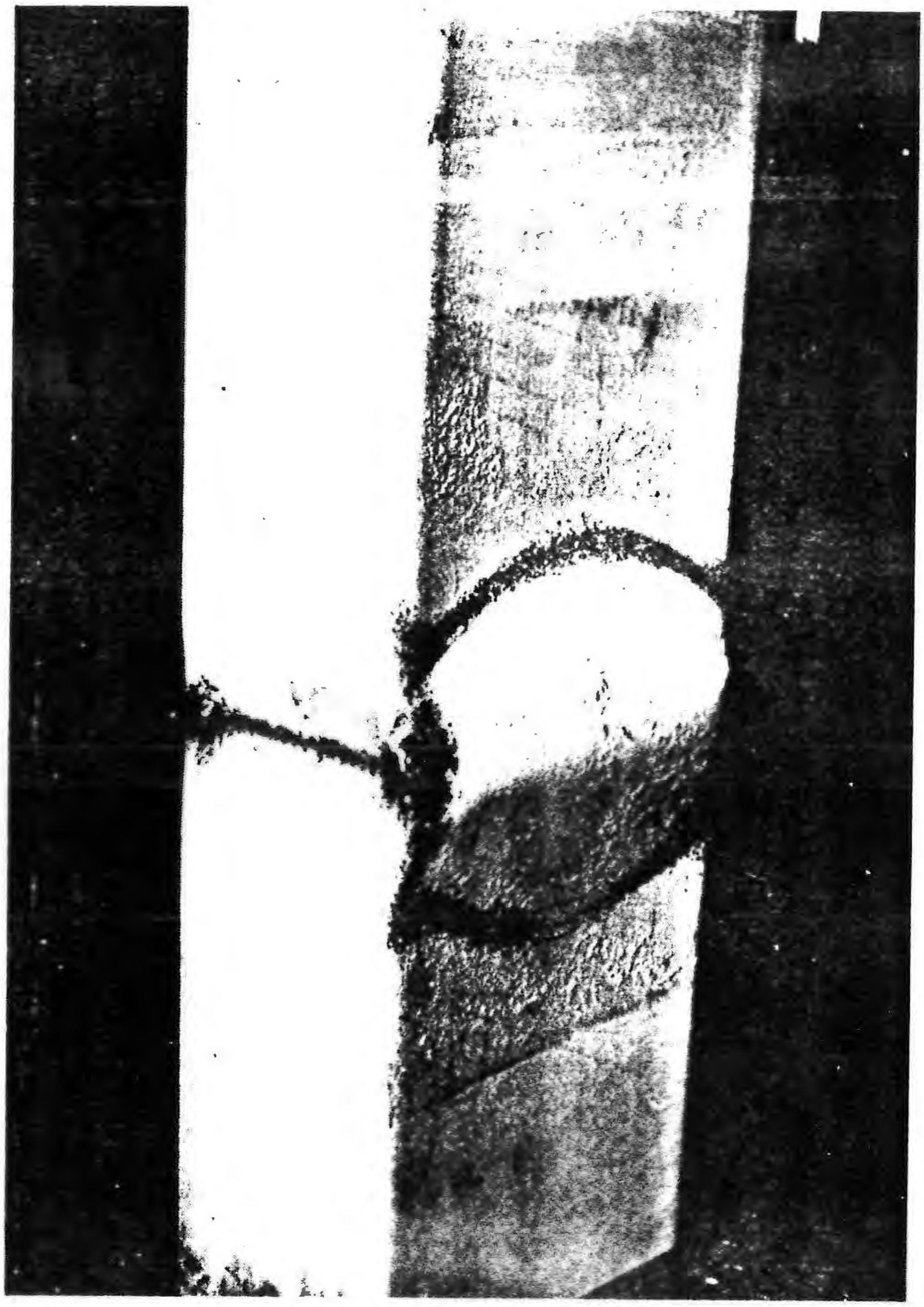


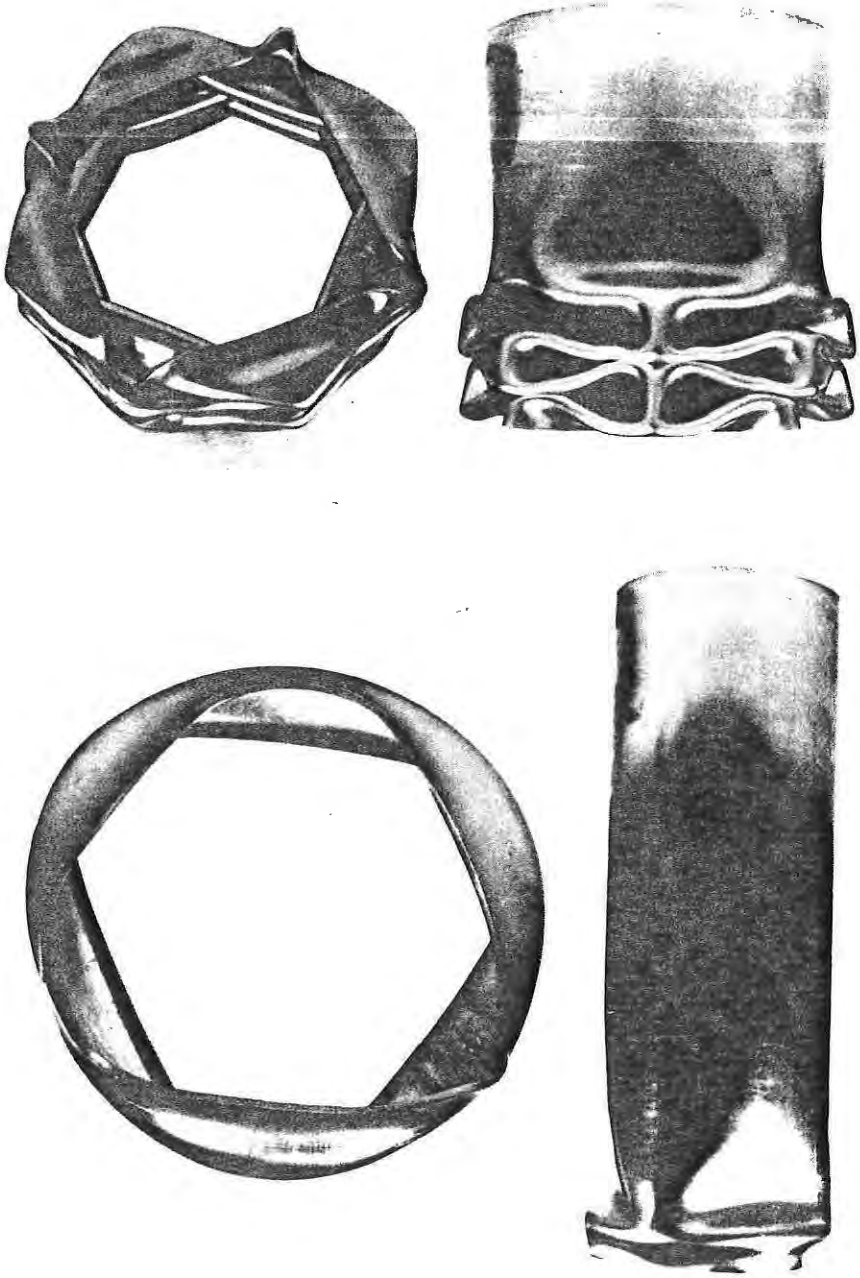


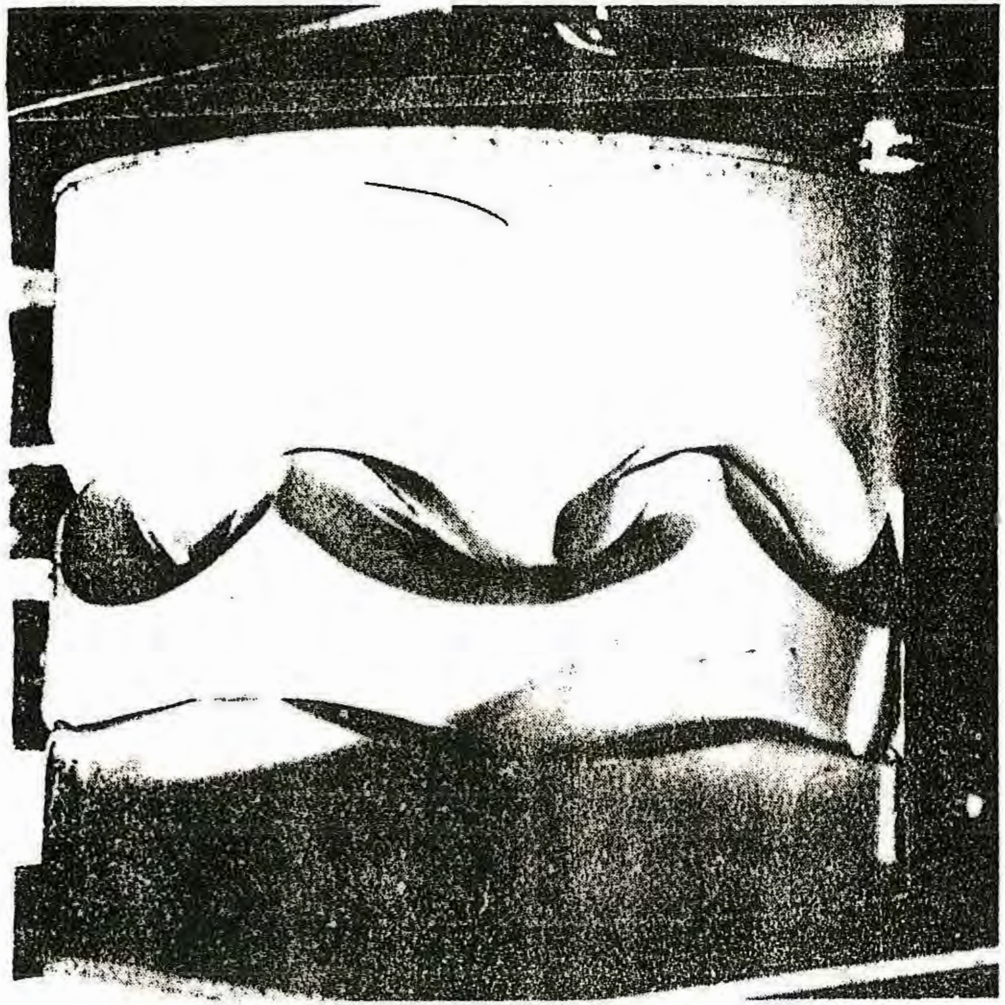
Le n° 138, un des 11 pylônes
détruits de la ligne 400 kV de Warande-Chevalet.

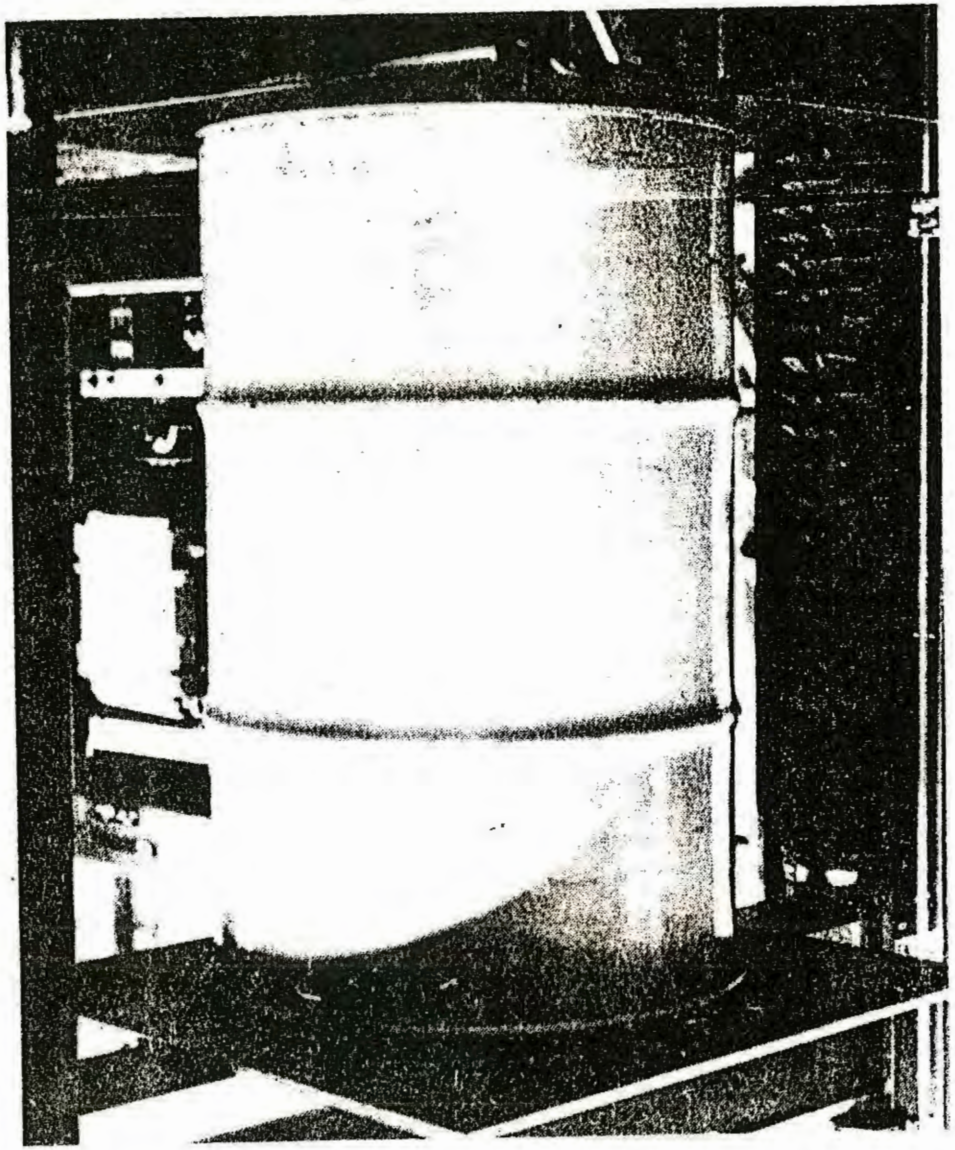






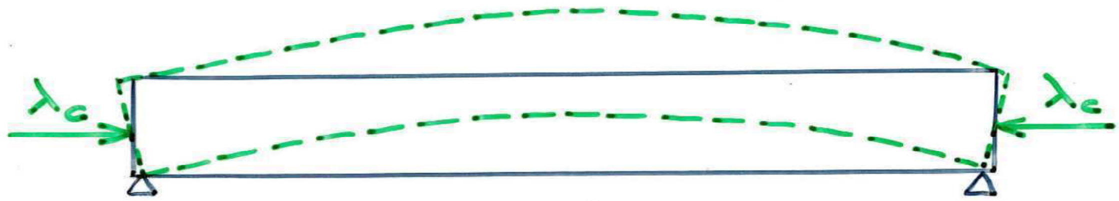


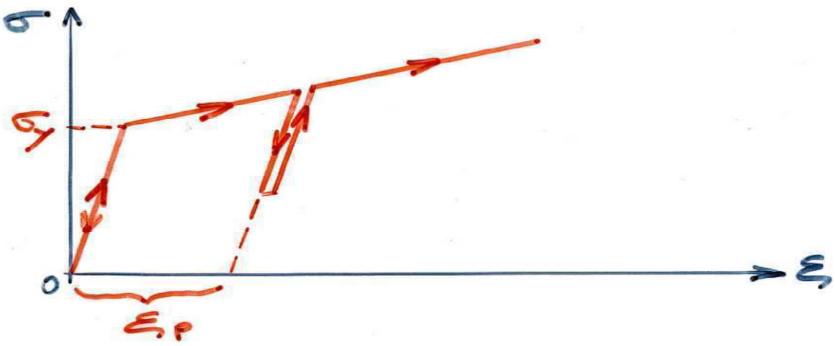




ELASTOPLASTIC BUCKLING AND POST-BUCKLING

THE CLASSICAL KNOWLEDGE





ENGESSER (1889)

* décharge exclue
* équilibre adjacent

$$\Longrightarrow \quad \lambda_c = \lambda_T \equiv \frac{\pi^2 E_T I}{\ell^2}$$

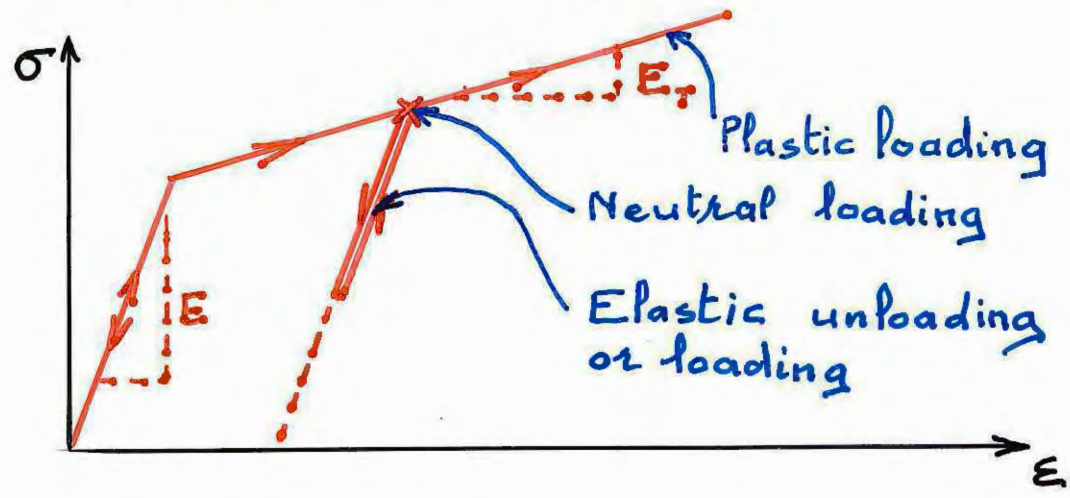
ENGESSER (1895), VON KARMAN (1910)

* prise en compte de la décharge
* équilibre adjacent

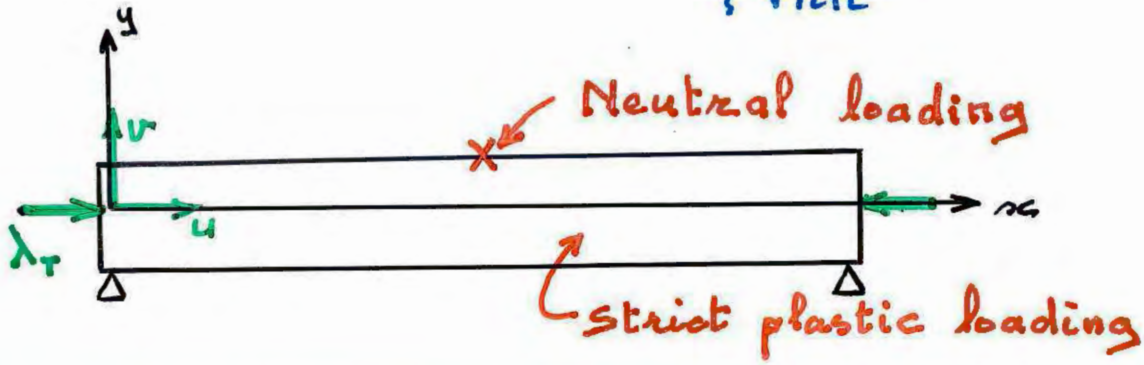
$$\Longrightarrow \quad \lambda_c = \lambda_R \equiv \frac{\pi^2 E_R I}{\ell^2}$$

PLASTIC BUCKLING

The constitutive law:

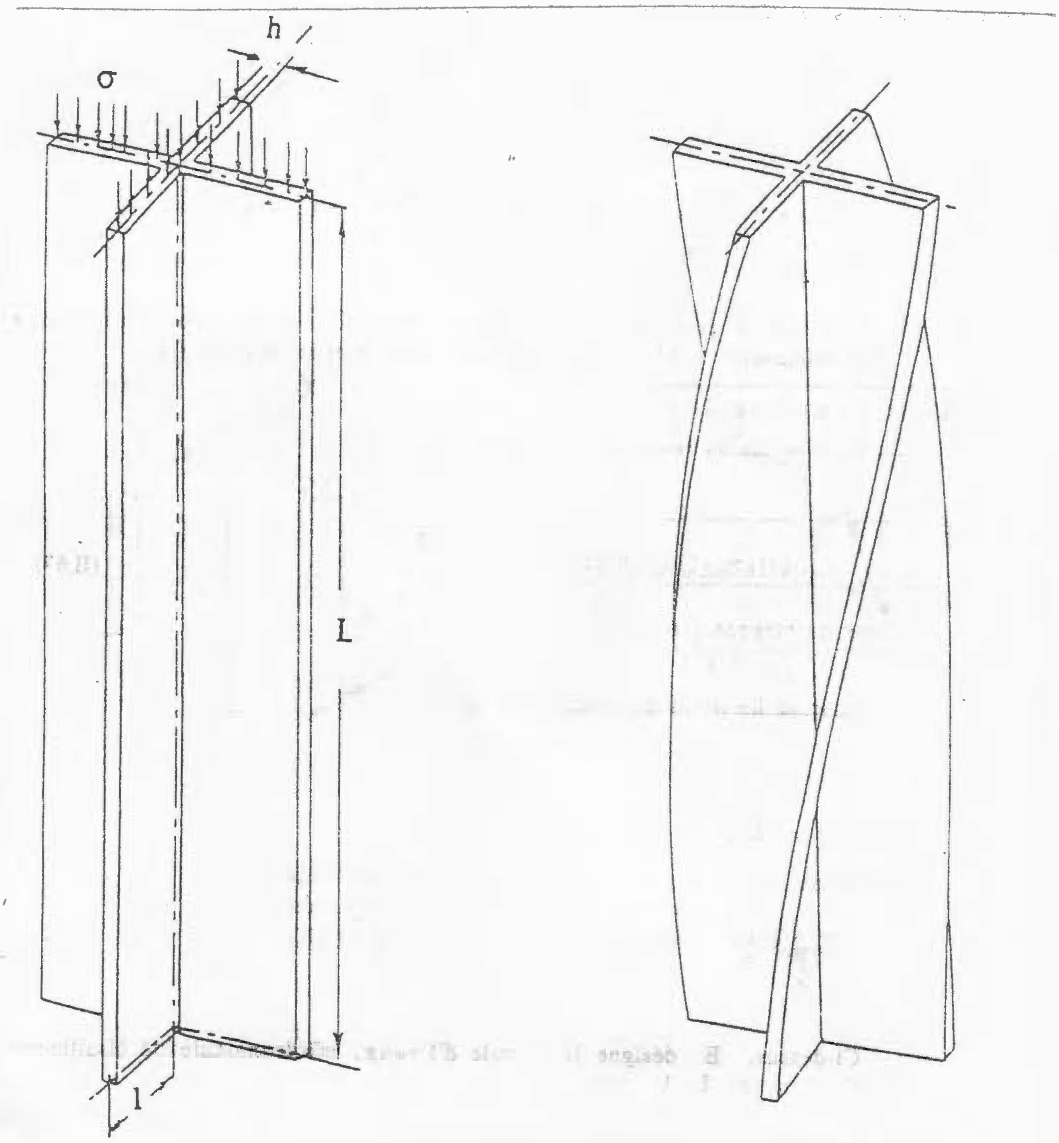


The buckling criterion: $\left\{ \begin{array}{l} \text{Shanley} \\ \text{Hill} \end{array} \right.$



* Occurs under increasing load even for symmetric structures
* At the bifurcation point, the initial velocity is characterized by a neutrally loaded point.

* Buckling occurs at the "critical load of the tangent modulus".
* A sufficient condition for uniqueness is given by the "elastic comparison solid".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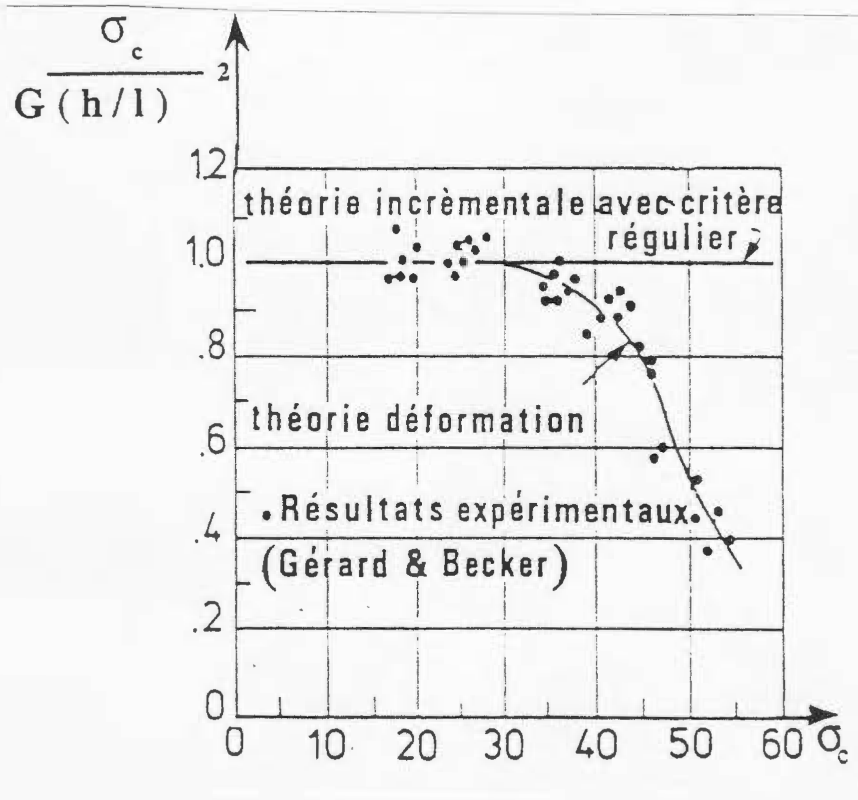
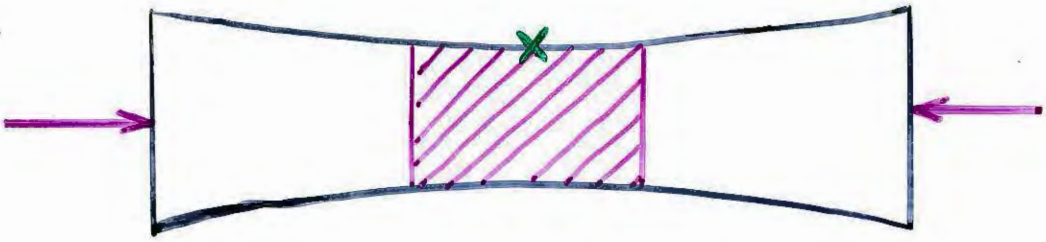
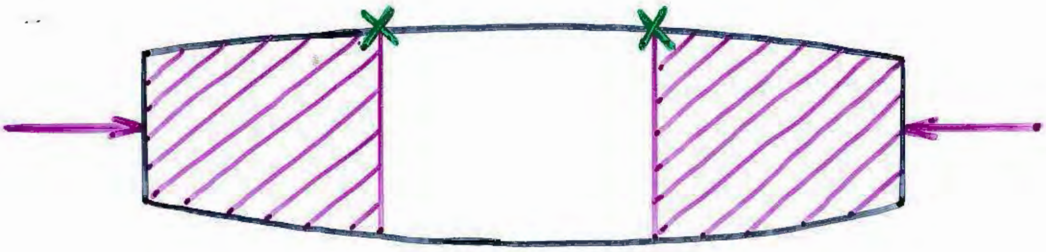
σ
L
l


$\dfrac{\sigma_c}{G(h/l)^2}$
1.2
1.0
.8
.6
.4
.2
0
théorie incrèmentale avec critère régulier
théorie déformation
• Résultats expérimentaux.
(Gérard & Becker)
0
10
20
30
40
50
60
σ_c

HETEROGENEOUS PREBIFURCATION CONDITIONS



(1)

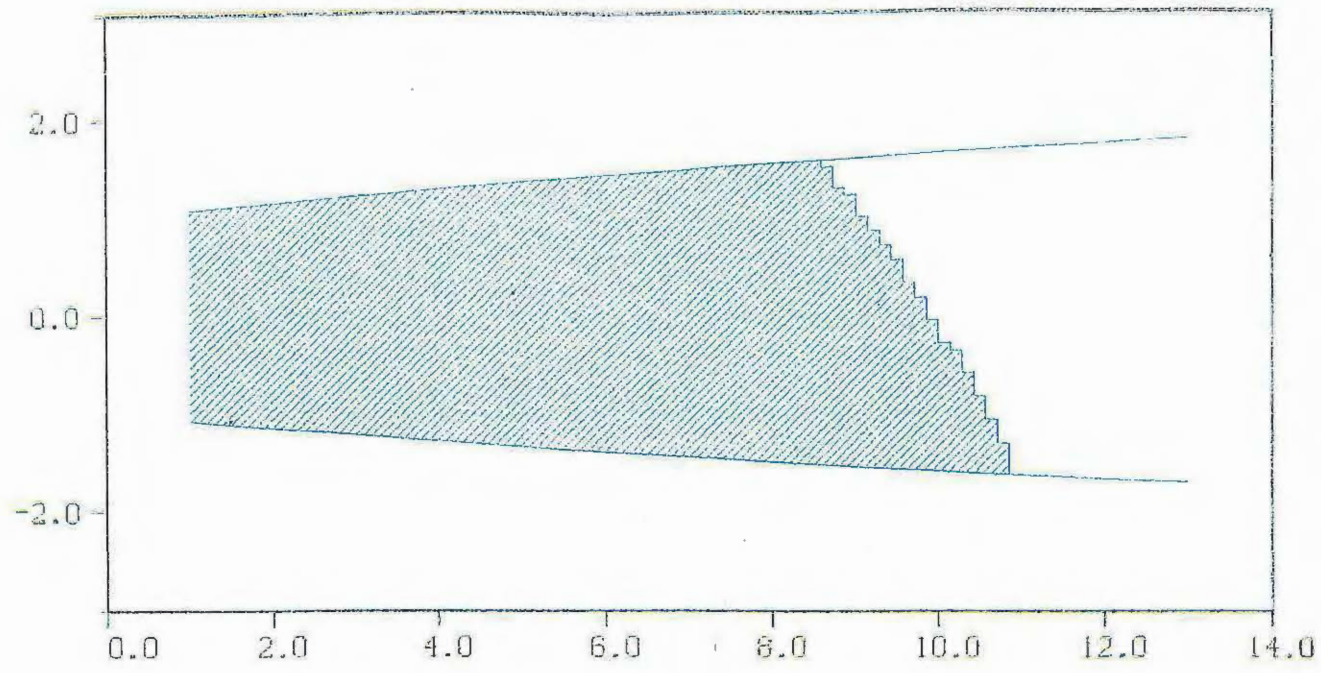


(2)

(3)

(1) and (3) : about the same as the homogeneous case.

(2) or (3) ? distinguished by the geometry of the structure.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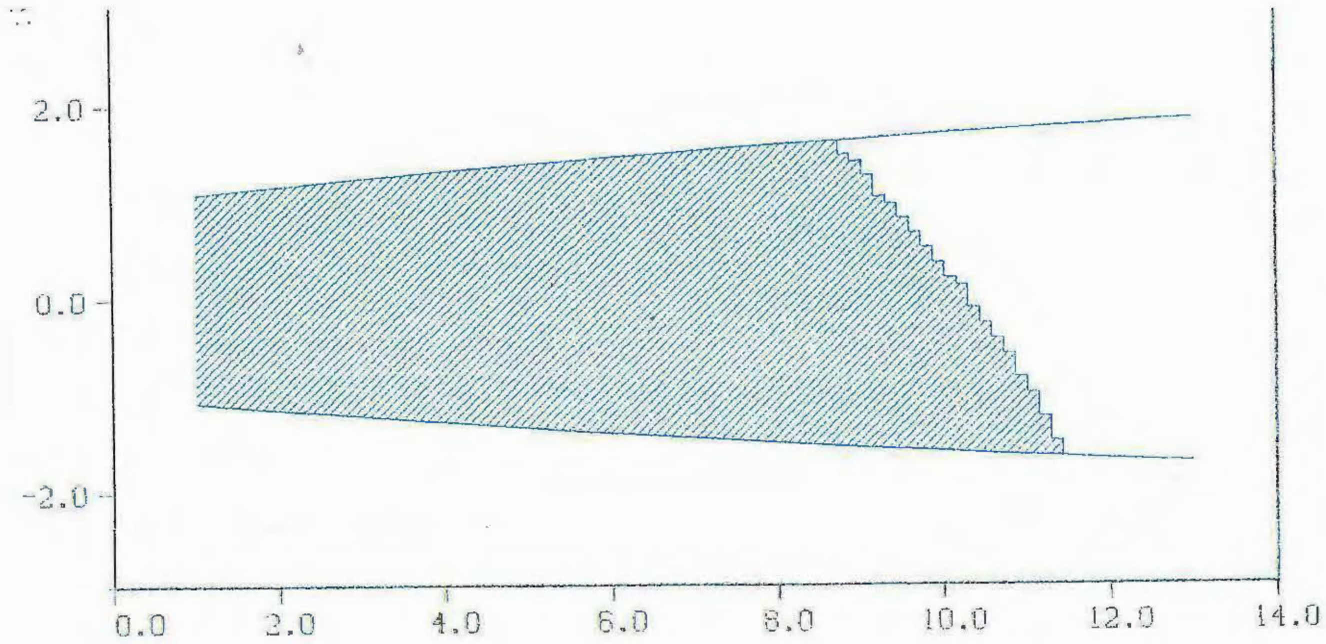
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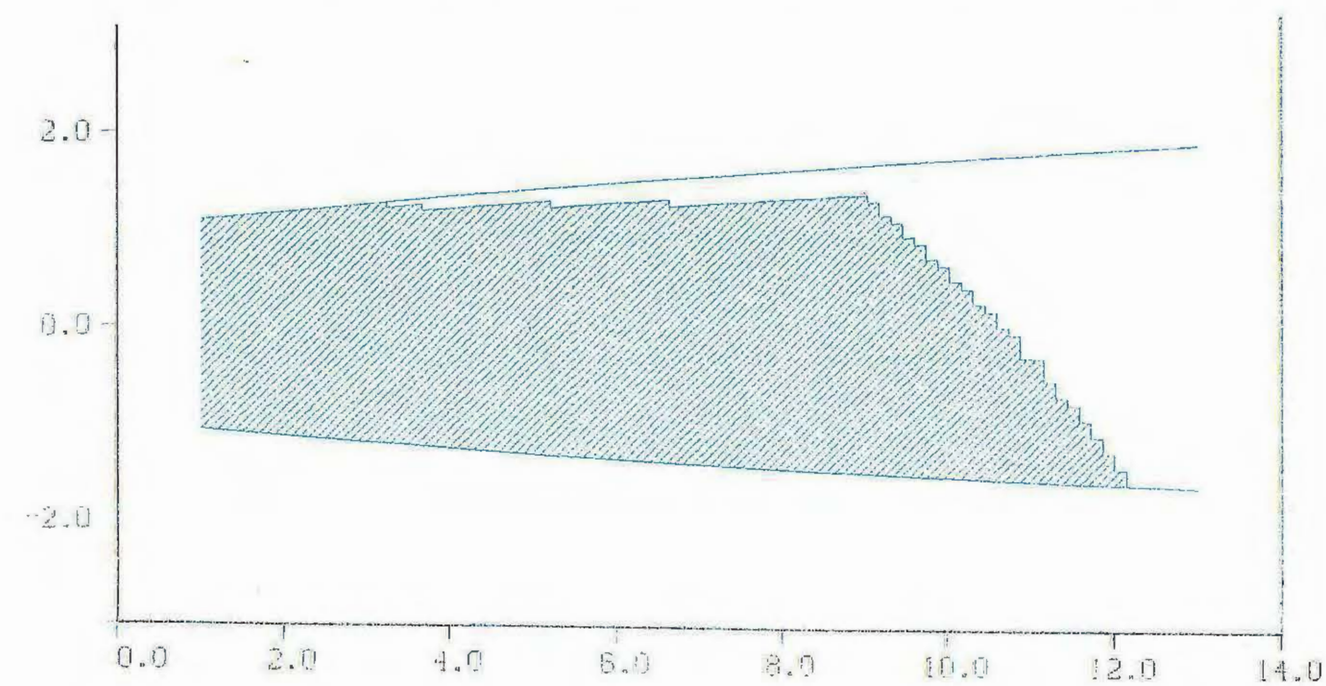
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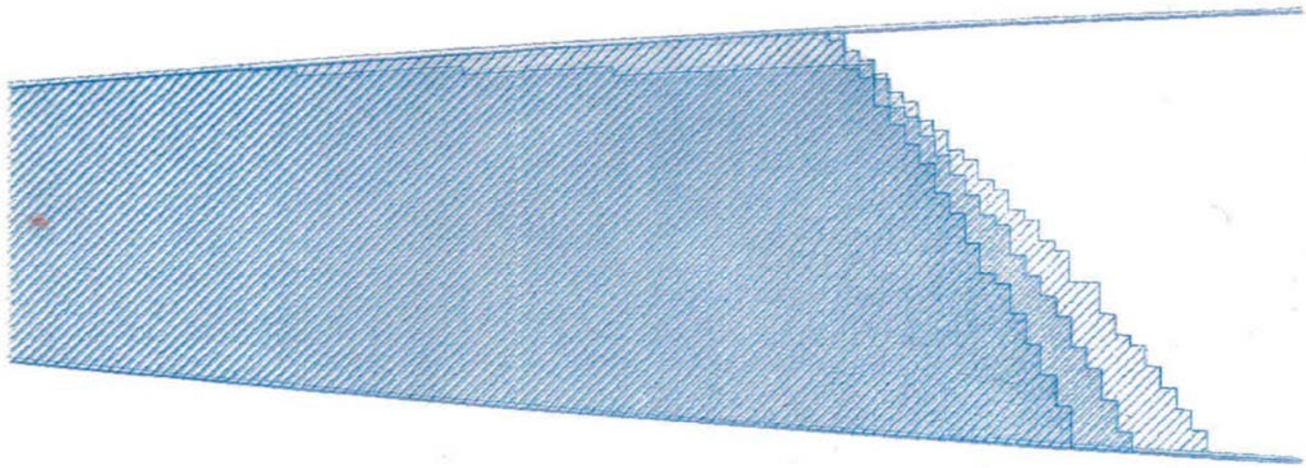
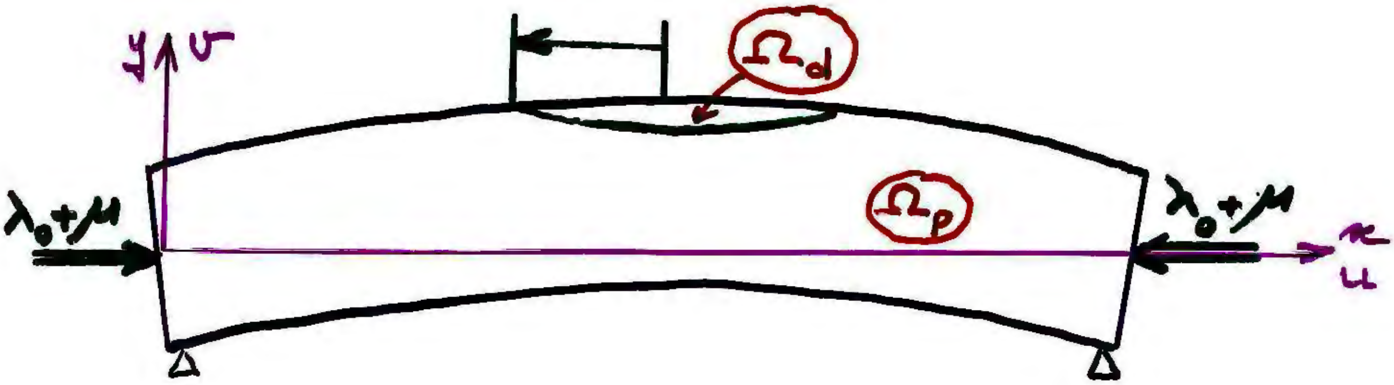
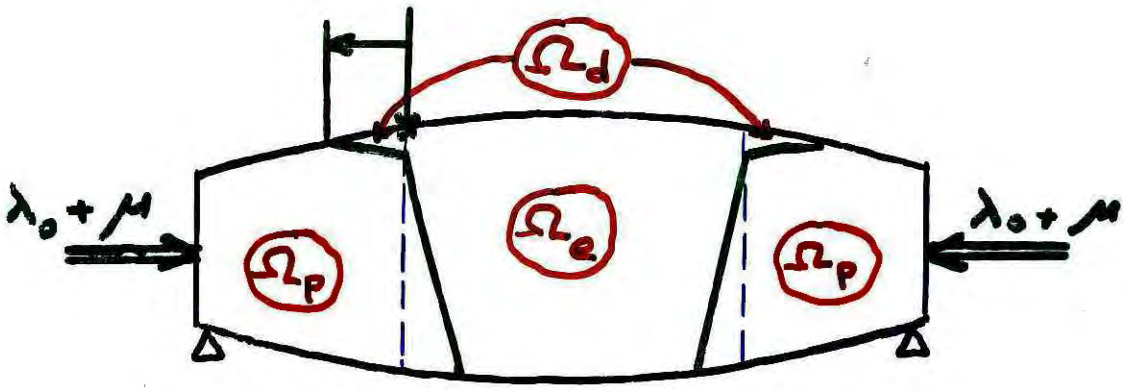


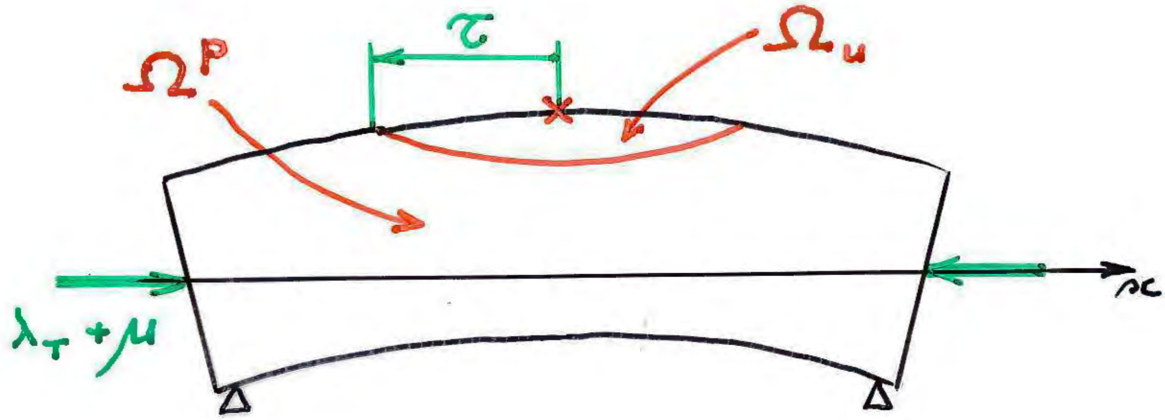




Inconnue	Mode	Vitesse initiale	Perturbation
$\dot{v}$	V	$\dot{v}^{o}$	v^{*}
$\dot{N}$	N	$\dot{N}^{o}$	N^{*}

POST BUCKLING

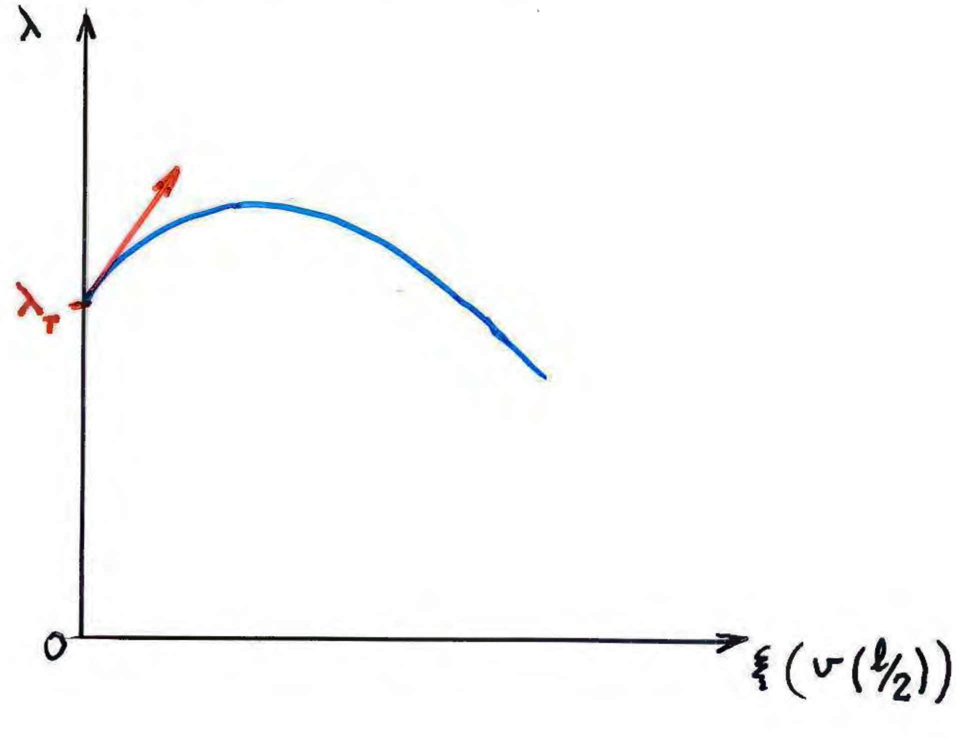
(J. Hutchinson)

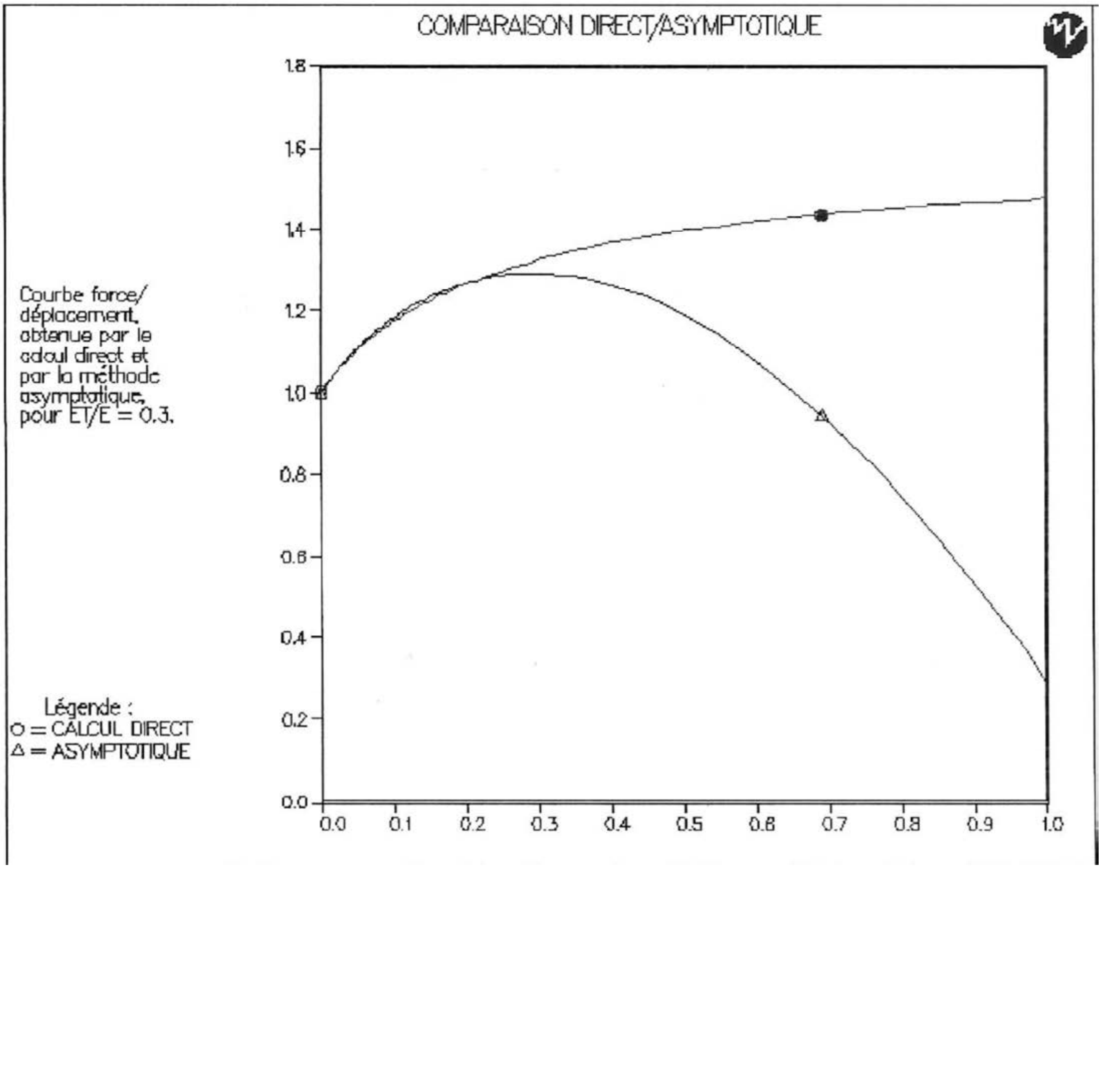


$$\mu = \text{Coef.}\, \tau^n + \text{h.o.t}$$

$$\lambda = \lambda_T + \lambda_1 \xi + \lambda_2 \xi^{1+\beta} + \dots$$

- $\lambda_1 > 0$
- $\lambda_2 < 0$
- $0 < \beta < 1$




COMPARAISON DIRECT/ASYMPTOTIQUE
Courbe force/
déplacement,
obtenue par le
calcul direct et
par la méthode
asymptotique,
pour ET/E = 0.3.
Légende :
O = CALCUL DIRECT
Δ = ASYMPTOTIQUE
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0