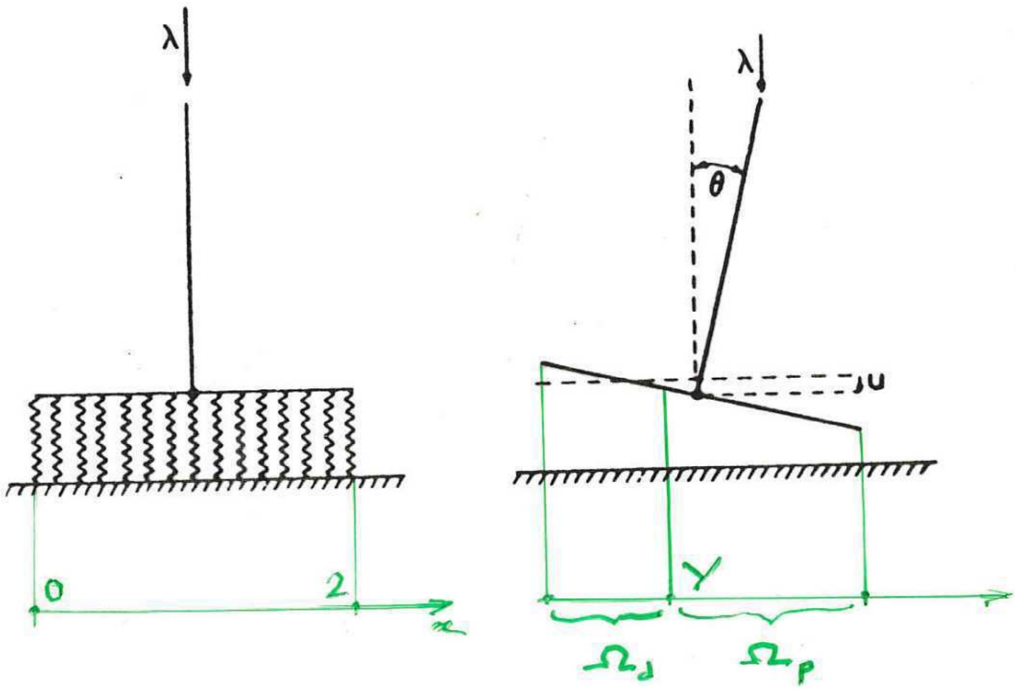
A SIMPLE MODEL



(1) $$\lambda = \int_0^2 \sigma(x)\,dx \,; \qquad \lambda L \theta = \int_0^2 (x-1)\,\sigma(x)\,dx$$

(2) $$\dot{\sigma} = E_T\,\dot{\varepsilon} \quad \text{ou} \quad \dot{\sigma} = E\,\dot{\varepsilon}$$

(3) $$\varepsilon(x,\lambda) = u(\lambda) + (x-1)\,\theta(\lambda).$$

Le problème d'équilibre (1), (2), (3) s'écrit :

$$\boxed{(\mathcal{P})\begin{cases}\dot{\lambda} + P(Y)\,\dot{\theta} = 0\\ \dot{\lambda}\theta + [\lambda - Q(Y)]\,\dot{\theta} = 0\\ Y \in [0,2]\end{cases}}$$

$$\begin{cases}P(Y) = 2E_T\left[\frac{1}{4}\frac{E-E_T}{E_T}Y^2 + Y - 1\right]\\ Q(Y) = \frac{E-E_T}{L}\left[-\frac{1}{6}Y^3 + \frac{1}{2}Y^2 + \frac{2E_T}{3(E-E_T)}\right]\end{cases}$$

Avec :

* $P(Y)$ a 1 seul zéro Y_R dans $[0,2]$, strictement positive dans $[0, Y_R[$, décroissante dans $[0, 2]$

* $Q(Y_R) = \lambda_R$ (Katchanov)

* $Q(Y)$ positive dans $[0,2]$, strictement croissante dans $]0,2[$,

* $Q(0) = \frac{2E_T}{3L} \equiv \lambda_T$

* $Q(2) = \frac{2E}{3L} \equiv \lambda_E$

Toute branche non triviale est transverse ($\dot{\theta} \neq 0$) en $\Theta = 0$ à la solution fondamentale.

$$(\mathcal{P}) \Rightarrow (\mathcal{P}^*) \begin{cases} \dot{\lambda} + P(Y)\,\dot{\theta} = 0 \\ P(Y)\,\theta + Q(Y) - \lambda \underset{def}{\equiv} \mathcal{F}(\theta, \lambda, Y) = 0 \end{cases}$$

Lemme 1 :

* $\exists !$ $Y(\theta, \lambda)$ satisfaisant $\mathcal{F}(\theta, \lambda, Y(\theta,\lambda)) = 0$
$\forall \theta \geqslant 0$ et $\lambda \in [\lambda_T, \lambda_E]$.

* $Y(\theta, \lambda)$ est une fonction analytique de ses arguments
$\forall \{\theta, \lambda\} \in \mathcal{V} \subset [0, +\infty[\times]\lambda_T, \lambda_E[$.

$$(\mathcal{P}^*) \Longrightarrow (\mathcal{P}^{**}) \quad \boxed{\begin{array}{l} \dfrac{d\lambda}{d\theta} = -P(Y(\theta, \lambda)) \\ \lambda(0) = \lambda_0 \end{array}}$$

THEOREME 1 :

* $(\mathcal{P})$ ne possède pas de solution transverse si $\lambda_0 \notin [\lambda_T, \lambda_E]$.

* $\forall \lambda_0 \in]\lambda_T, \lambda_E[$:

  * $\exists$ une solution transverse, avec décharge initiale sur $[0, Y_0]$, où $\lambda_0 = Q(Y_0)$.

  * $\exists !$ $\lambda(\theta)$, analytique sur $[0, +\infty[$, monotone,
    - strictement croissante si $Y_0 \in]0, Y_R[$ i.e. $\lambda_0 \in]\lambda_T, \lambda_R[$,
    - strictement décroissante si $Y_0 \in]Y_R, 2[$ i.e. $\lambda_0 \in]\lambda_R, \lambda_E[$,
    - constante si $Y_0 = Y_R$, i.e. $\lambda_0 = \lambda_R$

  et :

$$\begin{cases} Y(\theta) \xrightarrow[\theta \to +\infty]{} Y_R \\ \lambda(\theta) \xrightarrow[\theta \to +\infty]{} \lambda_R \end{cases}$$

CAS $\lambda(0) = \lambda_T$:

Soit une famille de zones de décharge initiale $\{[0, Y_n]\}$ avec $Y_n = \frac{1}{n}$.

$$\Rightarrow \lambda_n(0) = Q(1/n)$$

$$\Downarrow$$

famille de branches $\{\lambda_n(\theta)\}$.

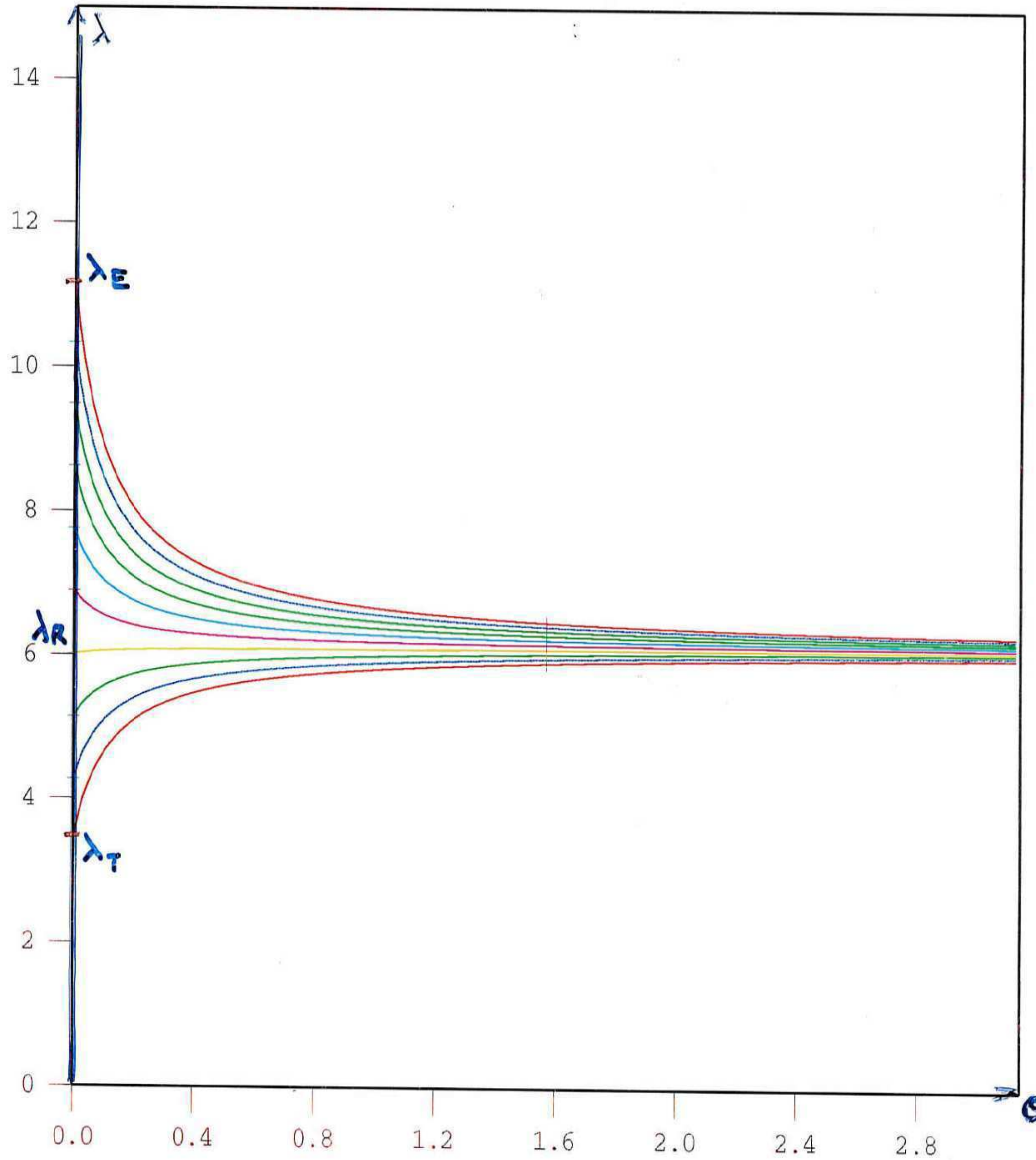
Lemme 2 :

La suite $\{\lambda_n(\theta)\}$ converge vers une branche $\lambda_\infty(\theta)$ solution de:

$$\begin{cases} \dfrac{d\lambda_\infty}{d\theta} = -P(Y(\theta, \lambda_\infty)) \\ \lambda_\infty(0) = \lambda_T \end{cases}$$

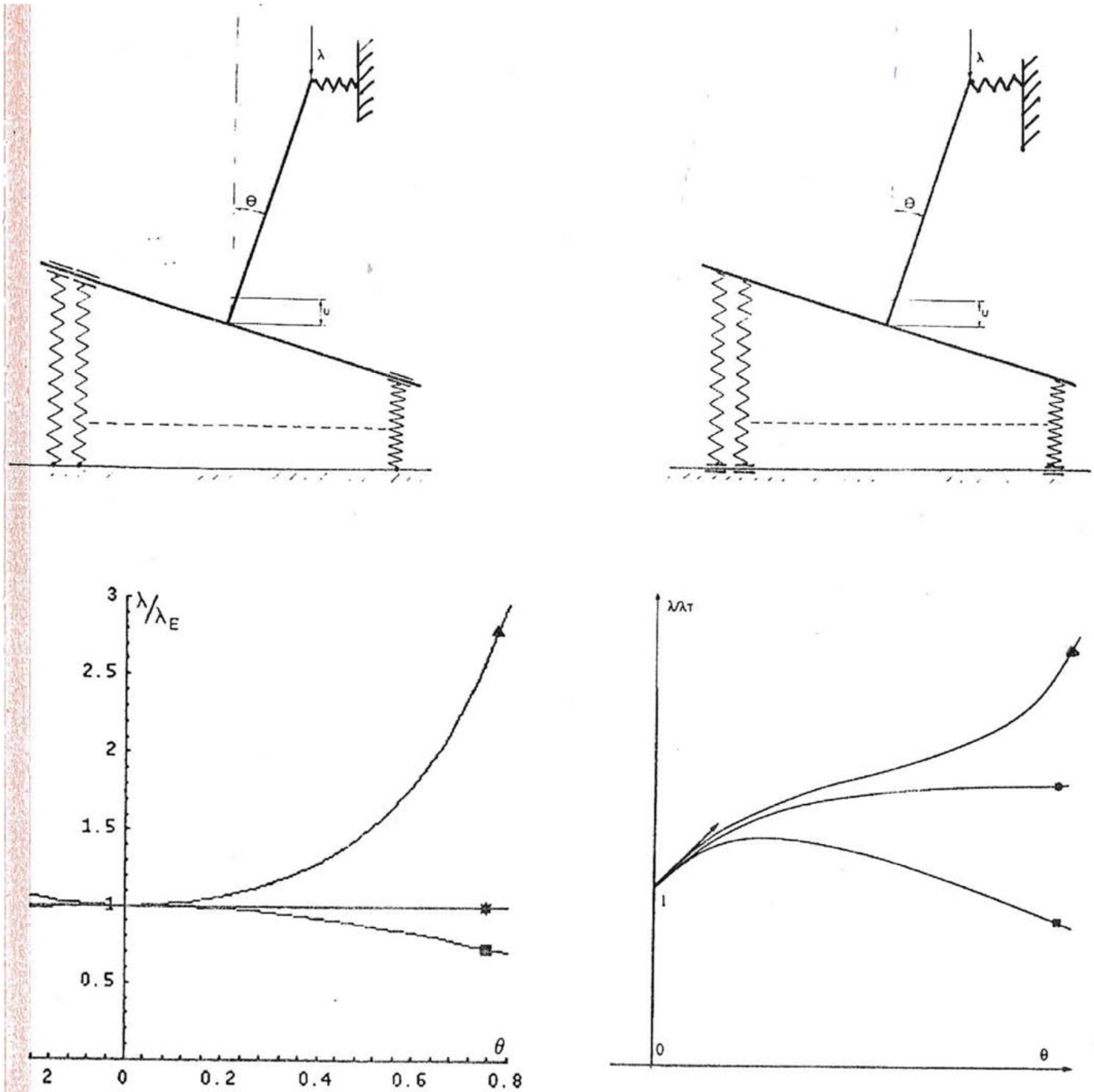
THÉORÈME 2 :

$(\mathcal{P})$ possède (pour $\dot{\theta} \geq 0$) une branche bifurquée unique $\theta(\lambda)$ vérifiant $\theta(\lambda_T) = 0$. Celle-ci présente une zone de décharge initiale réduite au point $Y = 0$, et est analytique en $\sqrt{\lambda - \lambda_T}$ pour λ assez voisin de λ_T.



λ
14
12
λE
10
8
λR
6
4
λT
2
0
0.0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
θ

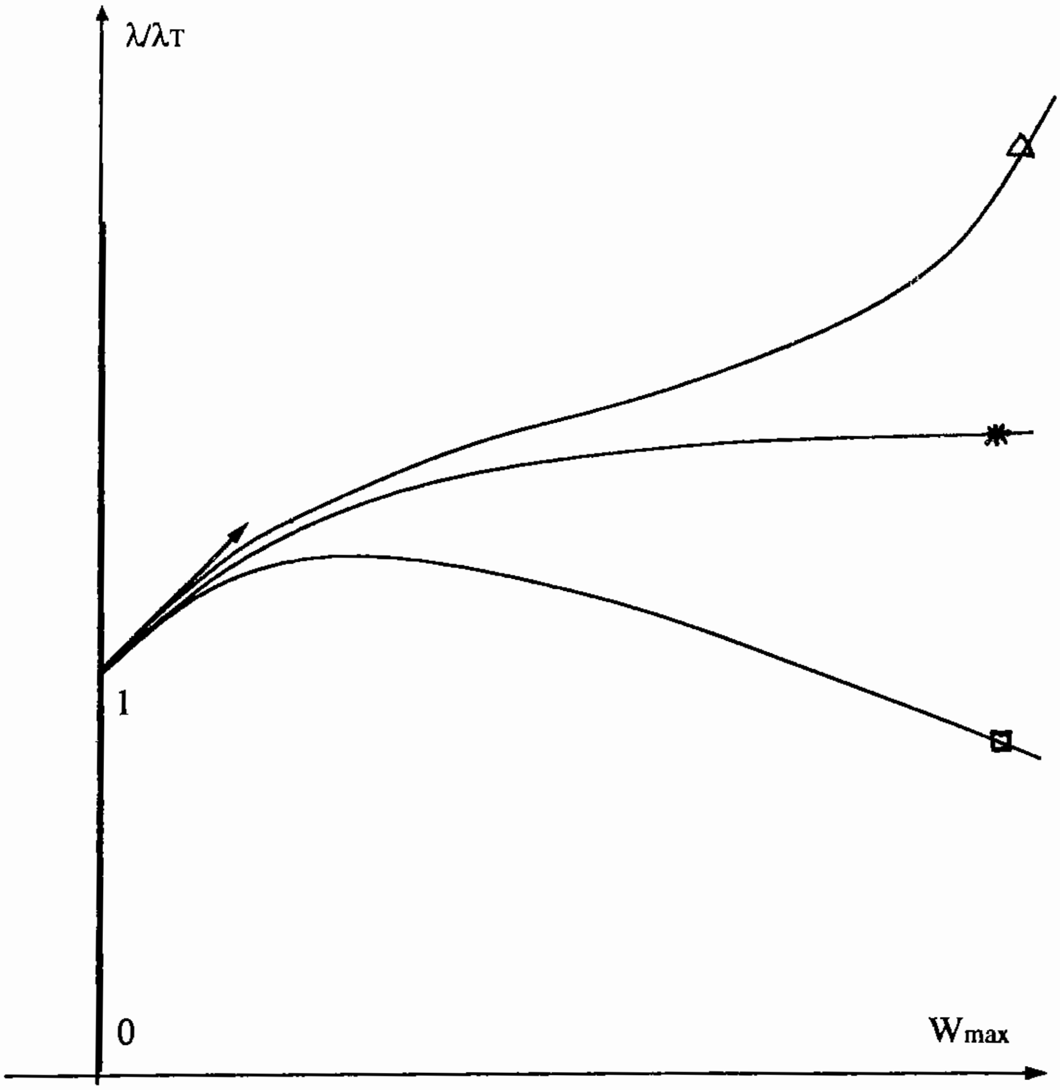
EFFECT OF GEOMETRICAL NONLINEARITIES



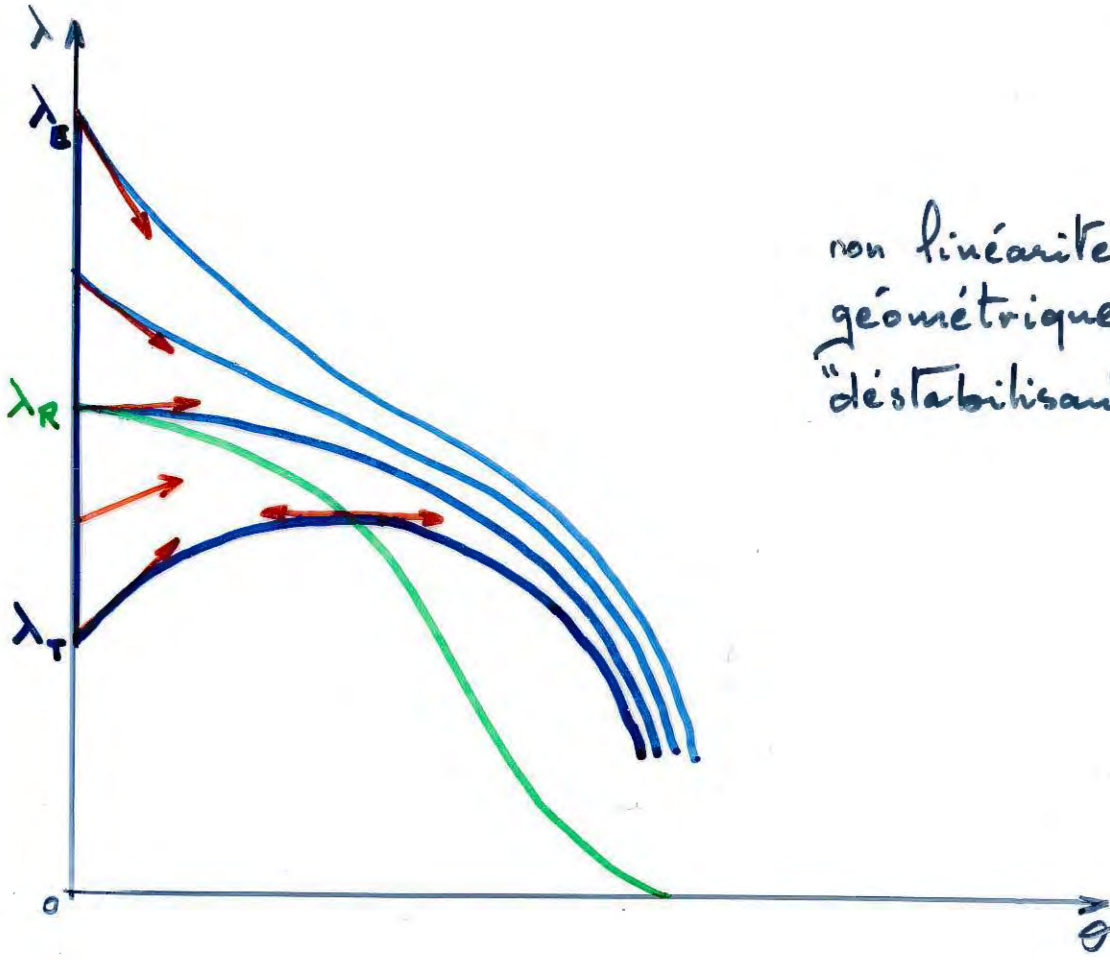
a - Comportement élastique linéaire b - comprtement élastoplastique

✱ : géométriquement linéaire; ▲ : non-linéarité stabilisante;

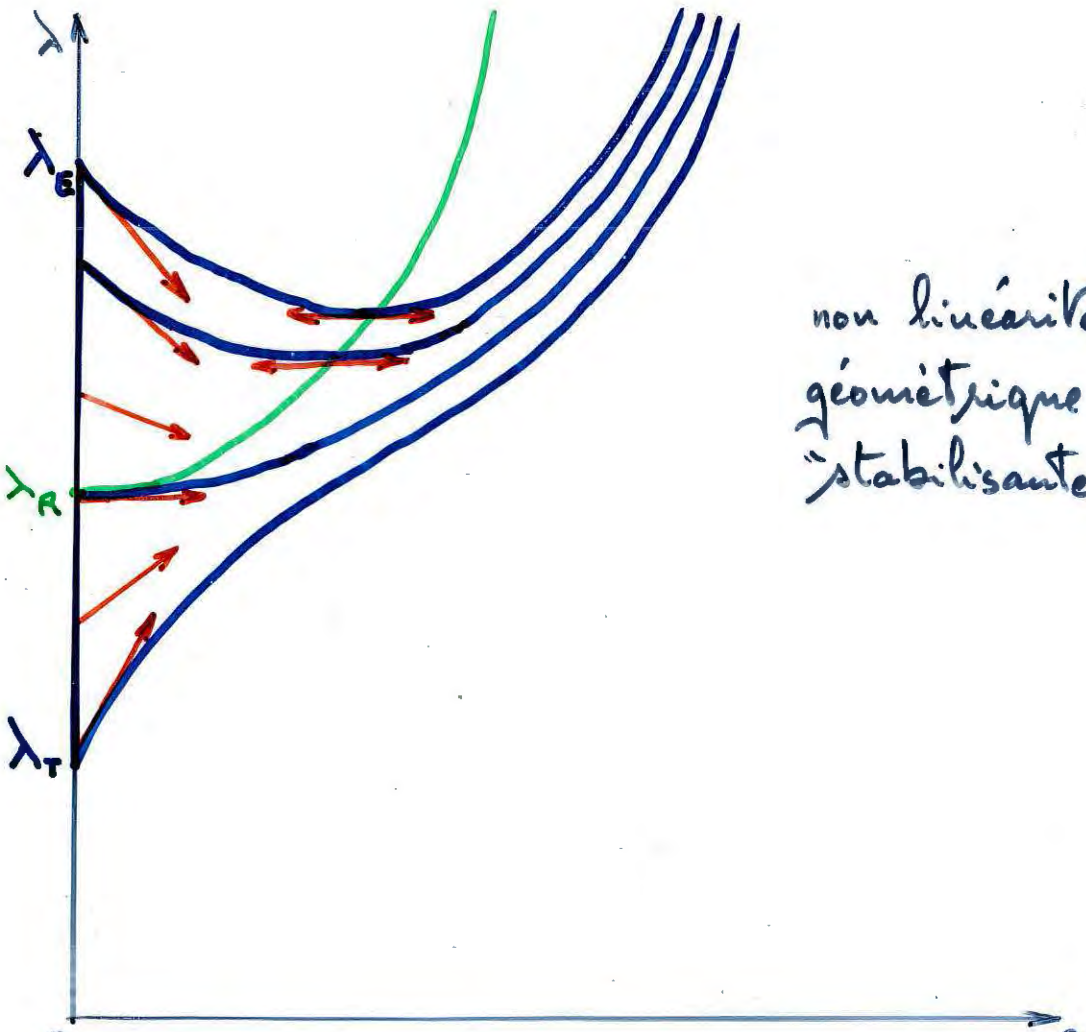
■ : non-linéarité déstabilisante.



λ/λT
1
0
Wmax

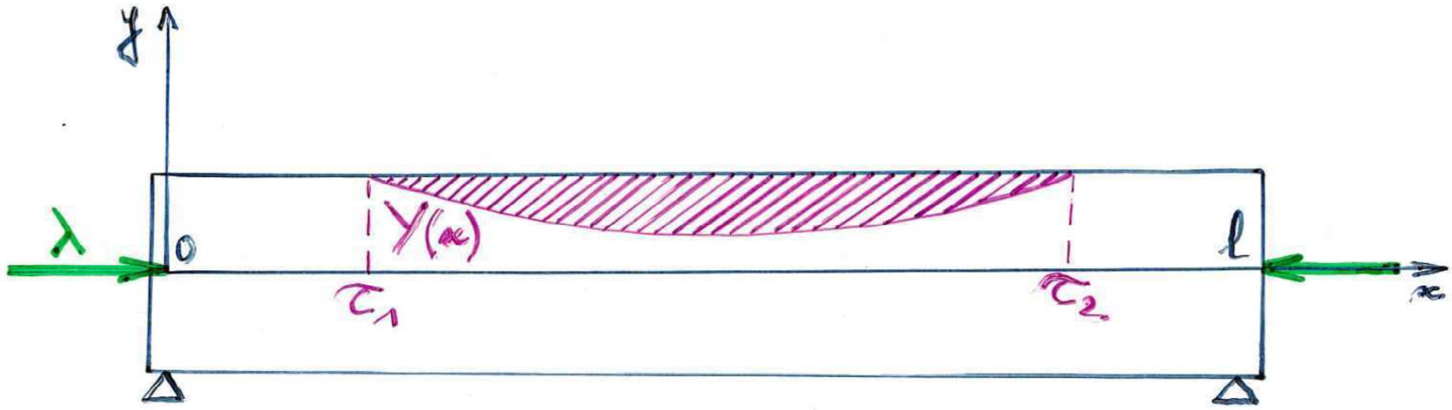


non linéarité géométrique "déstabilisante"



non linéarité géométrique "stabilisante"

THE ELASTOPLASTIC COMPRESSED BEAM



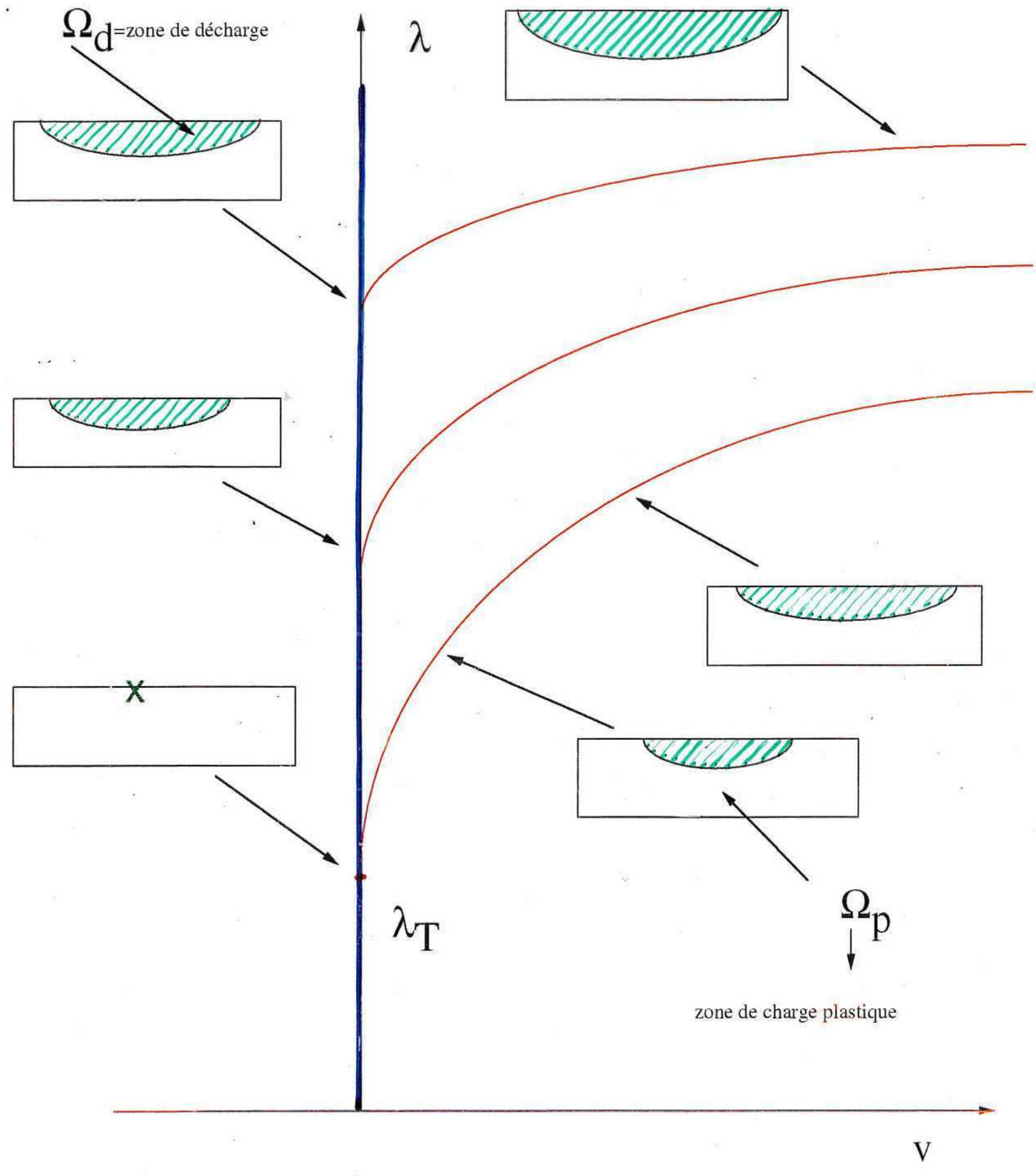
$$\begin{cases} P(Y)\,\dot{v}'' + \dot{\lambda} = 0 \\ Q(Y)\,\dot{v}'' + \dot{\lambda}\, v + \lambda \dot{v} = 0 \end{cases} \quad x \in \,]\tau_1, \tau_2[$$

$$\Longrightarrow \begin{cases} Q(x)\,\dot{v}'' + \lambda \dot{v} + \dot{\lambda}\, v = 0 & x \in \,]0, \ell[\\ \dot{v}(0) = \dot{v}(\ell) = 0 \end{cases}$$

Where :

$$Q(x) = \begin{cases} E_T I & \text{if plastic loading} \\ E I & \text{if elastic loading or unloading} \\ Q(Y(x)) & \text{if both loading and unloading with a boundary } Y(x). \end{cases}$$

A FREE BOUNDARY PROBLEM



THE INITIAL VELOCITY PROBLEM

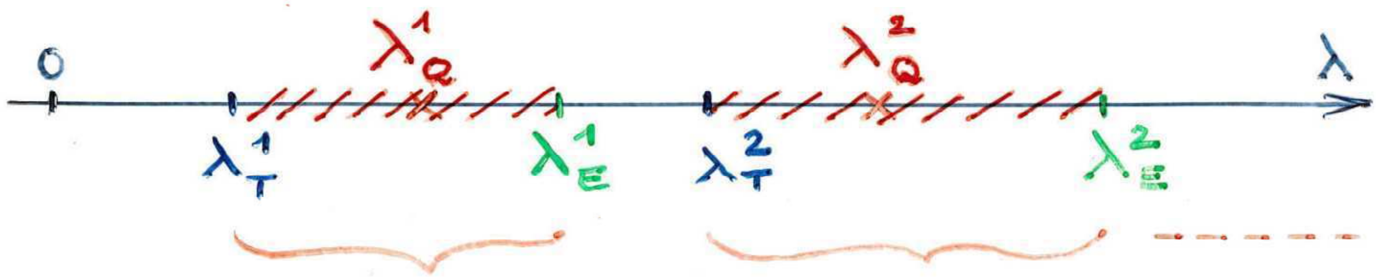
* $\{\lambda \in \mathbb{R},\ v(x) \equiv 0\}$ is the trivial branch

* Consider 3 auxiliary problems:

$$\left.\begin{array}{l} \dot{v}'' + \dfrac{\lambda}{E_T I}\dot{v} = 0 \\[2ex] \dot{v}'' + \dfrac{\lambda}{Q(x)}\dot{v} = 0 \\[2ex] \dot{v}'' + \dfrac{\lambda}{EI}\dot{v} = 0 \end{array}\right\} \quad \dot{v}(0) = \dot{v}(\ell) = 0$$

Each of these problems have a strictly increasing infinite sequence of eigenvalues, with

$$\lambda_T^n \leq \lambda_Q^n \leq \lambda_E^n, \ \forall n$$



THEOREM 1:

* $\forall \lambda \in]\lambda_T^1, \lambda_R^1[$, $\exists !$ initial velocity transverse to the fundamental branch.

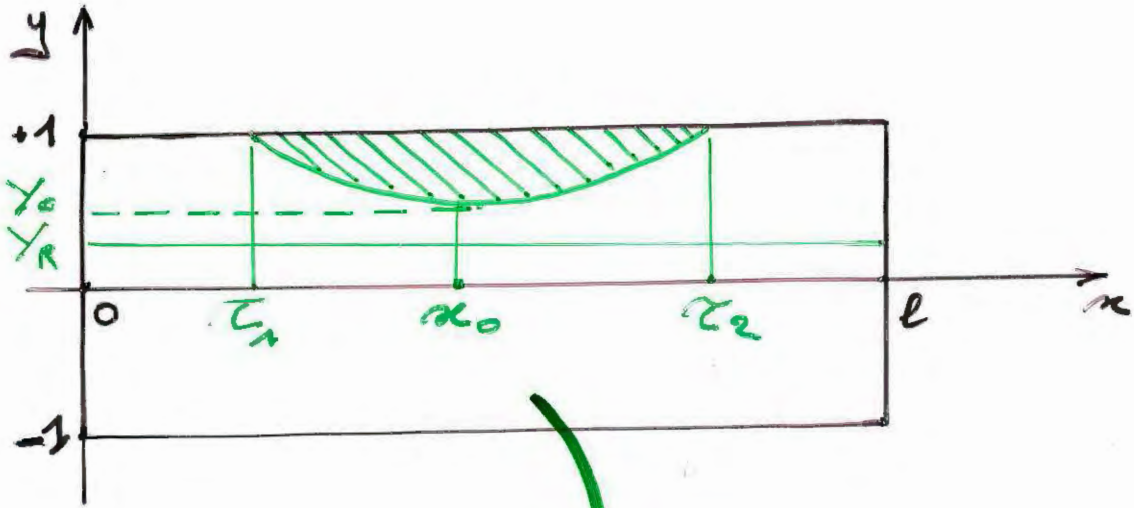
* This velocity involves an unloading zone of strictly positive measure, and occurs under increasing load.

* There exists an explicit correspondance between the value of the load, and the size of the unloading zone.

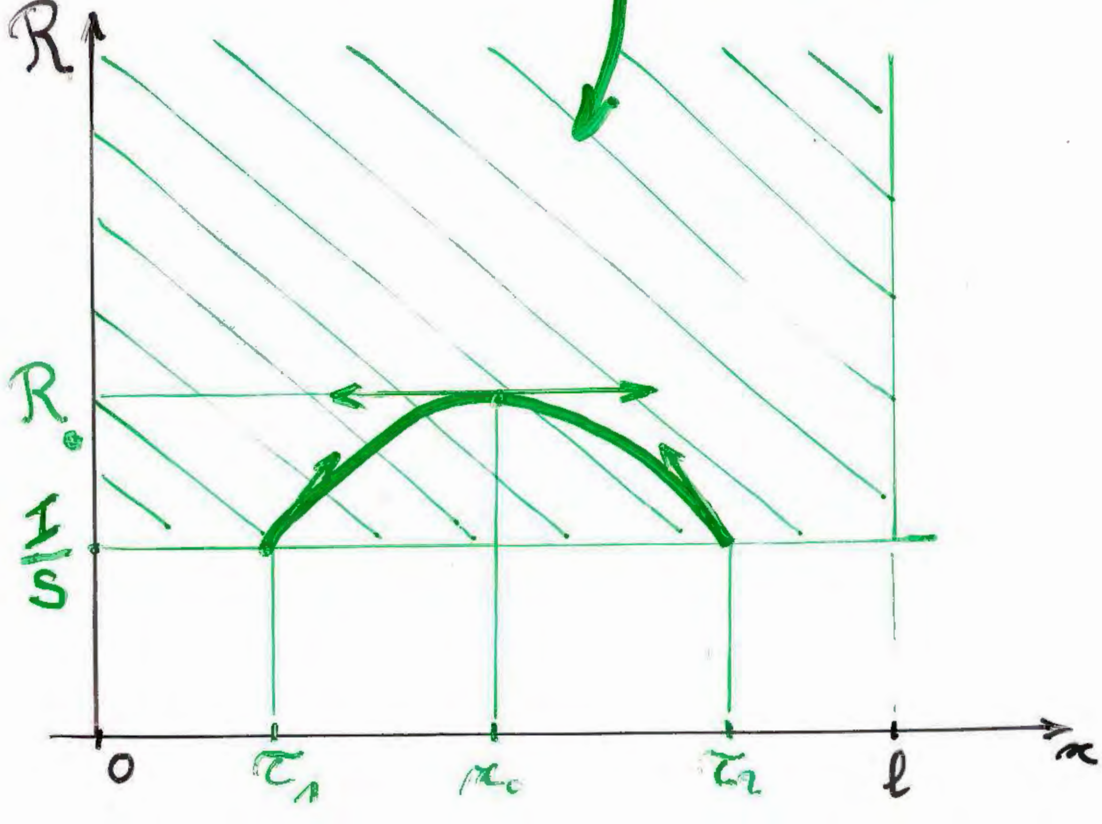
THEOREM 2:

Let $\lambda \in [\lambda_T^n, \lambda_E^n]$ for any n. Then $\{\lambda, 0\}$ is a bifurcation point of the initial problem, and there is no bifurcation point outside of these intervals.

Sketch of the proof



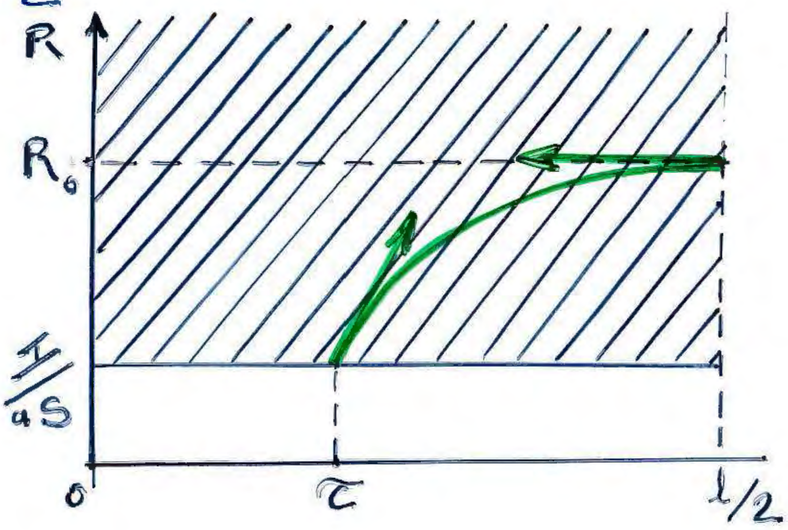
$$\mathcal{R}(x) \equiv \mathcal{R}(Y(x)) = \frac{Q(Y)}{P(Y)}$$



Sketch of the proof:

Based on a shooting problem for unknown $R(x)$.

* Assume the solution is symmetric w.r. to $\ell/2$. Then:



$$\begin{cases} R'' + \dfrac{\lambda}{P(f(R))} = 0 \\ R(\ell/2) = R_0, \quad R'(\ell/2) = 0, \; R(\tau) = \dfrac{1}{S} \\ R'(\tau) = \dfrac{1}{S}\sqrt{\dfrac{I}{E_T I}}\, \mathrm{cotg}\sqrt{\dfrac{\lambda}{E_T I}}\,\tau \end{cases}$$

$$\boxed{\Phi(\lambda, R_0) = 0} \Rightarrow \text{unique solution}$$

* There is no other solution.

The symmetric case:

$$\boxed{\begin{aligned} & \mathcal{R}'' + \frac{\lambda}{P(f(\mathcal{R}))} = 0 \\ & \mathcal{R}(\ell/2) = \mathcal{R}_0 , \ \mathcal{R}'(\ell/2) = 0, \ \mathcal{R}(\tau) = \frac{I}{S} \\ & \mathcal{R}'(\tau) = \frac{1}{S}\sqrt{\frac{I}{E_T I}} \operatorname{cotg} \sqrt{\frac{\lambda}{E_T I}}\, \tau \end{aligned}}$$

$$\tau = \ell/2 - \frac{\mathcal{X}(\mathcal{R}_0)}{\sqrt{\lambda}}, \quad \tau \in]0, \ell/2[$$

$$\Phi(\mathcal{R}_0) = 0$$

where:

$$\mathcal{X}(\mathcal{R}_0) = \int_{I/S}^{\mathcal{R}_0} \frac{ds}{\sqrt{2\int_s^{\mathcal{R}_0} \frac{d\eta}{P(f(\eta))}}}$$

$$\Phi(\mathcal{R}_0) = \frac{I}{S}\sqrt{\frac{\lambda}{E_T I}} \operatorname{cotg}\left(\frac{\lambda}{E_T I}\frac{\ell}{2} - \frac{\mathcal{X}(R_0)}{\sqrt{E_T I}}\right) - \sqrt{2\int_{I/S}^{R_0} \frac{d\eta}{P(f(\eta))}}$$

$$\Rightarrow \begin{cases} \mathcal{X}(R_0) \text{ and } \Phi(R_0) \text{ are continuous} \\ \mathcal{X}(R_0) \xrightarrow[R_0 \to \infty]{} \ell/2\sqrt{\lambda_R} \end{cases}$$

$$0 \leq \left(\sqrt{\lambda} - \sqrt{\lambda_T}\right) \ell/2 \leq \mathcal{X}(R_0) \leq \sqrt{\lambda}\, \ell/2$$

$\exists$ R_0^1 and $\mathcal{R}_0^2$ in $]I/S, +\infty[$: $\begin{cases} \mathcal{X}(\mathcal{R}_0^1) = \ell/2\left(\sqrt{\lambda} - \sqrt{\lambda_T}\right) \\ \mathcal{X}(R_0^2) = \ell/2\sqrt{\lambda} \end{cases}$

Sketch of the proof
(continued)

$$\Phi(R_0^1) < 0$$
$$\Phi(R_0^2) > 0$$

Since $\Phi(R_0)$ is continuous
$\Phi(R_0, \lambda) = 0$ has a solution.

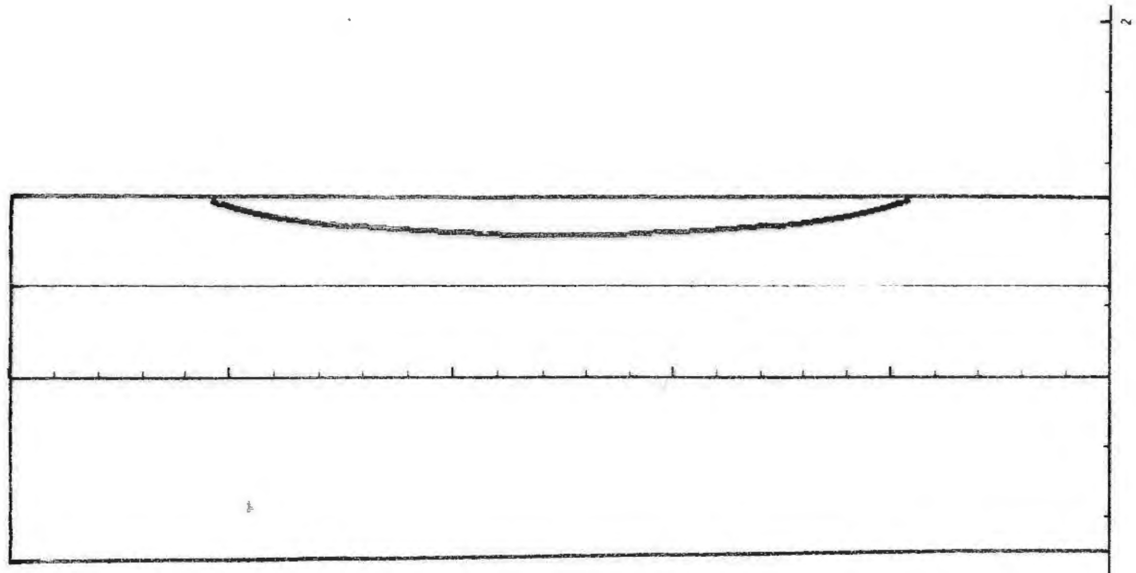
$\Rightarrow$ The existence is proved

$\Rightarrow$ The uniqueness is elementary:
R_0 and $\tilde{R}_0$ being 2 solutions
such that $R_0 > \tilde{R}_0$ then
$R(x) > \tilde{R}(x)$, corresponding
to $\dot{v}(x)$ and $\tilde{\dot{v}}(x)$:

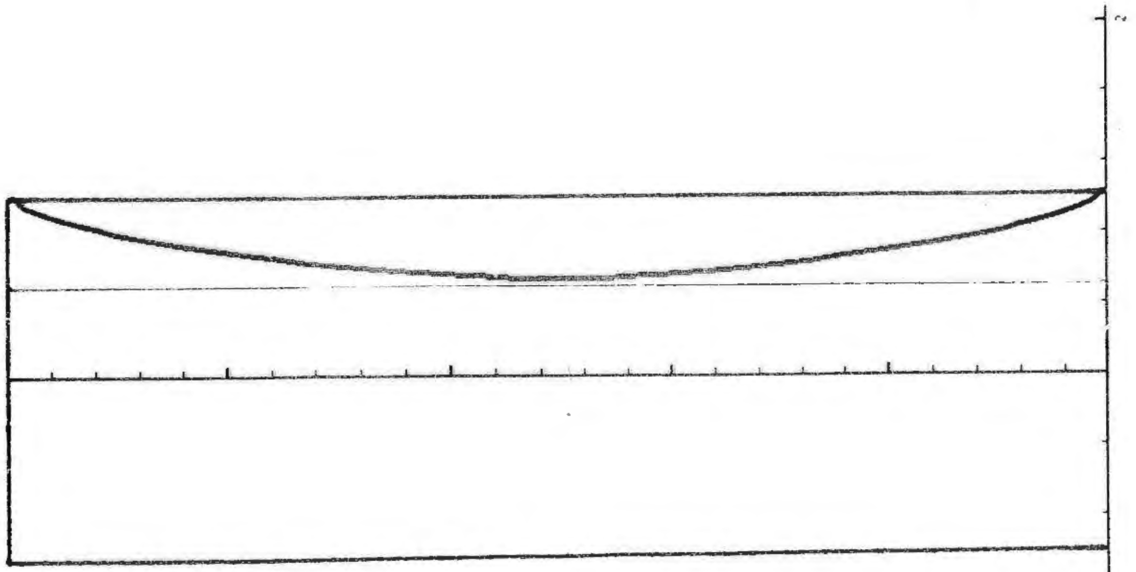
$$\begin{cases} \dot{v}'' + \dfrac{\lambda}{Q(f(R))}\dot{v} = 0 \\ \dot{v}(0) = \dot{v}(\ell) = 0 \end{cases} ; \quad \begin{cases} \tilde{\dot{v}}'' + \dfrac{\lambda}{Q(f(\tilde{R}))}\tilde{\dot{v}} = 0 \\ \tilde{\dot{v}}(0) = \tilde{\dot{v}}(\ell) = 0 \end{cases}$$

Impossible since Q and f are monotonous

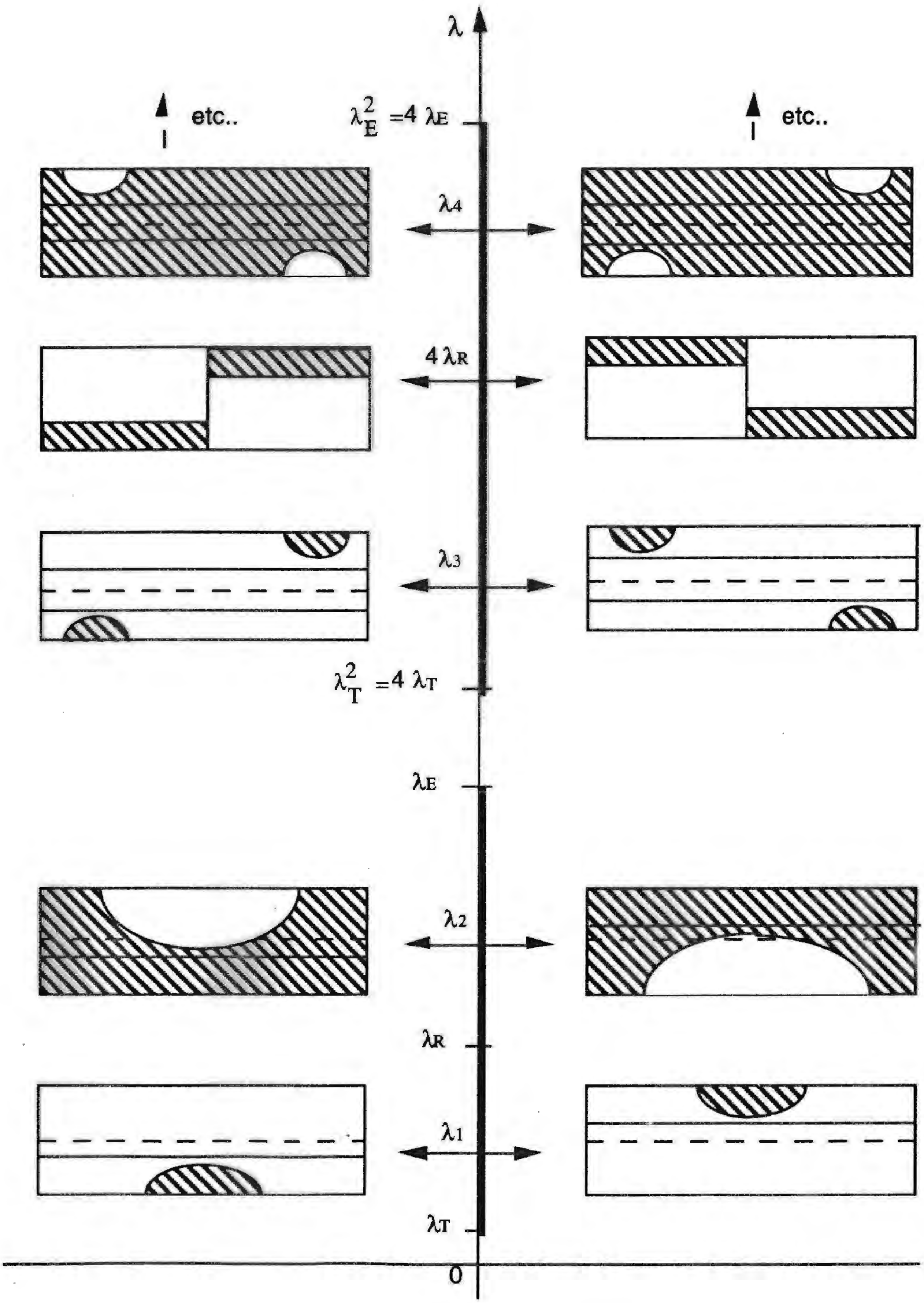
There is no non symmetric solution.



$$\lambda = \lambda_T + 0.05(\lambda_R - \lambda_T)$$

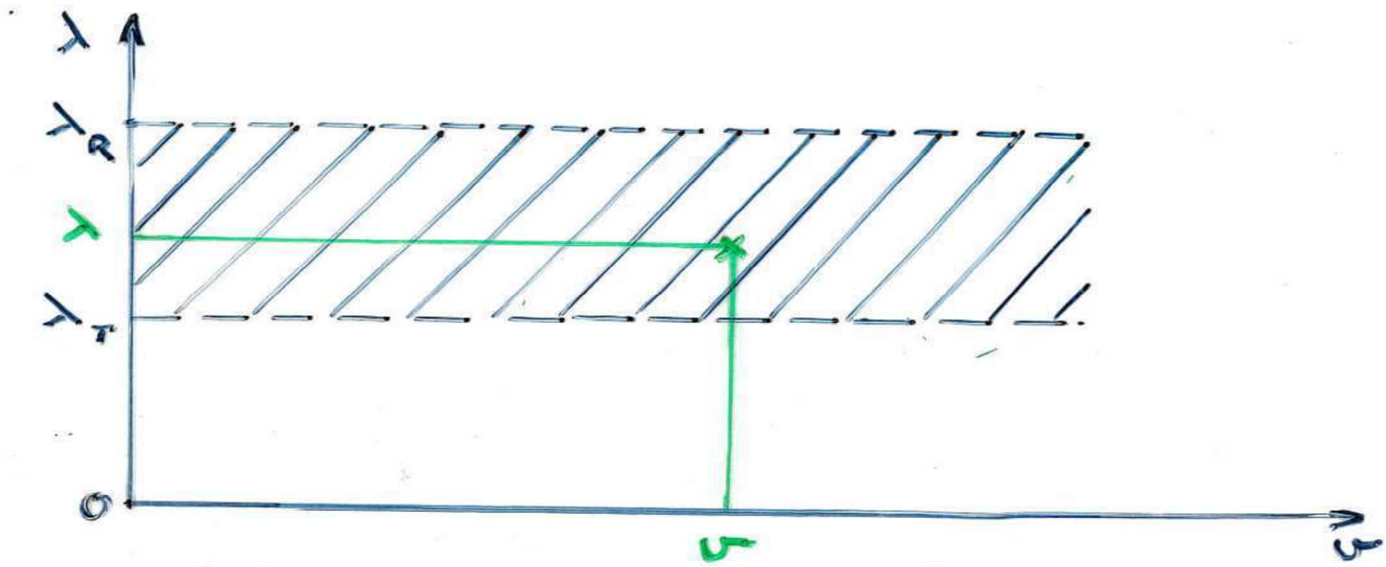


$$\lambda = \lambda_T + 0.6(\lambda_R - \lambda_T)$$



Allure des zones de décharge

TOWARDS A GLOBAL BIFURCATION AND CONTINUATION METHOD

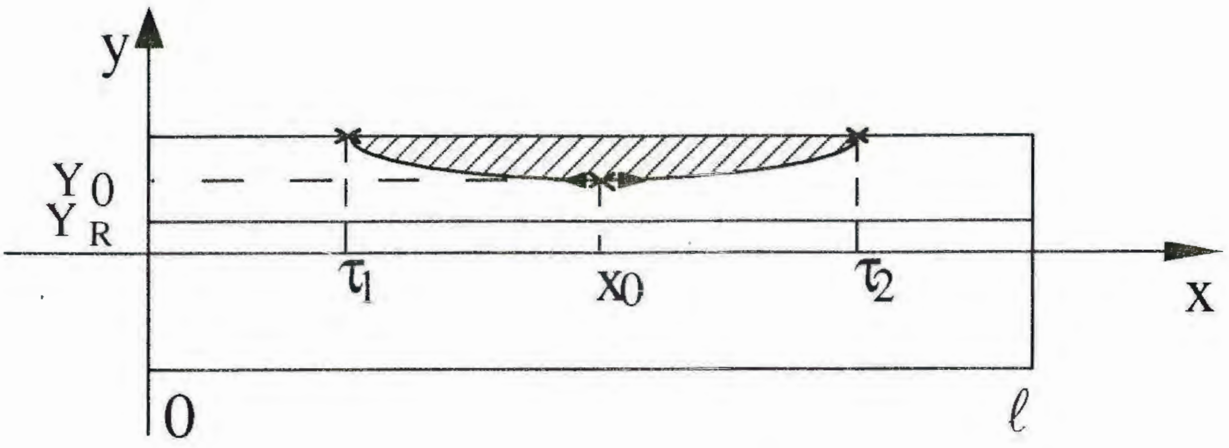


$$\begin{cases} R'' + \dfrac{\lambda}{P(f(R))} - \boxed{\upsilon''} = 0 \\ R(\ell/2) = R_0 \quad , \quad R'(\ell/2) = 0 \\ R'(\tau) = \underline{\hspace{6cm}} \end{cases}$$

Not $\Phi(\lambda, R_0) = 0$

but :

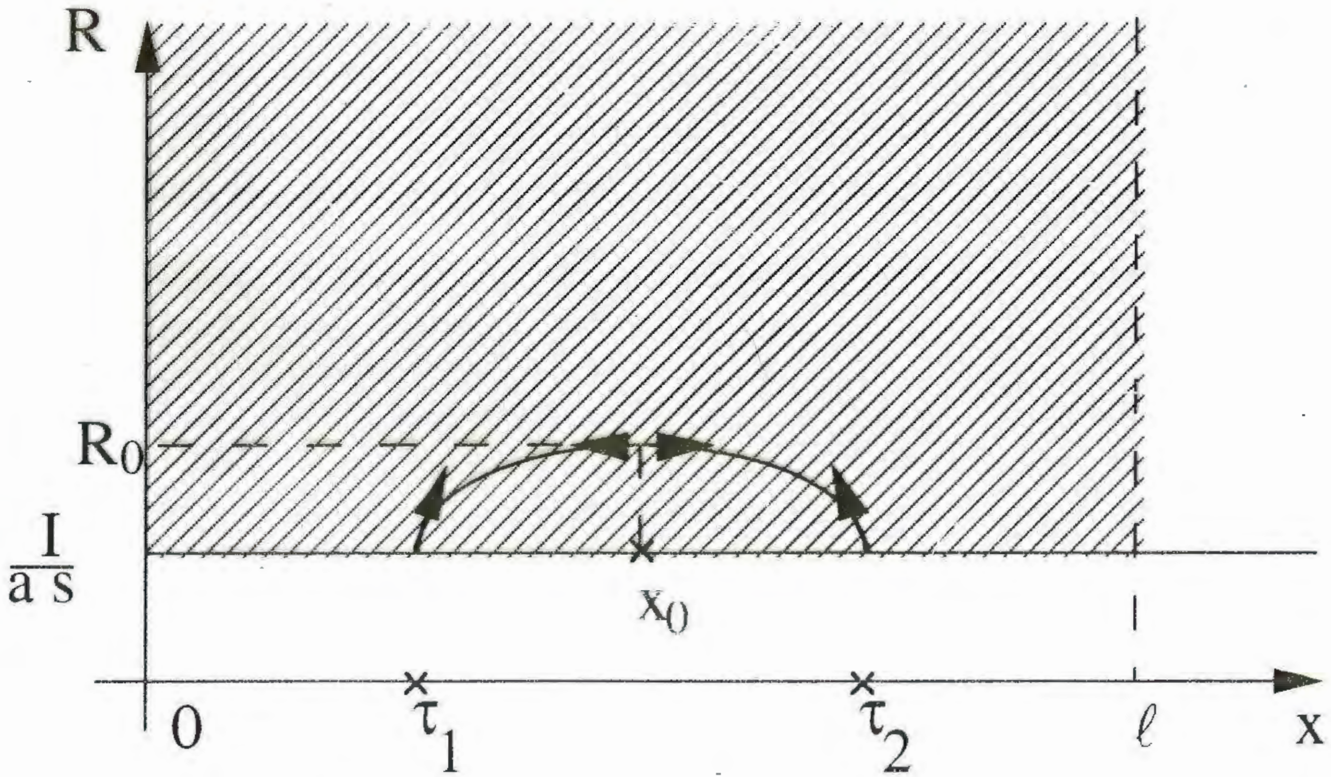
$$\boxed{\Phi(\lambda, \upsilon, R_0) = 0}$$



Let $R(x) = \dfrac{Q(Y(x))}{P(Y(x))} = R(Y(x))$

Then:

$$\begin{cases} * \begin{cases} R'' + \dfrac{\lambda}{P(f(R))} - v'' = 0 & \text{in }]\tau_1, \tau_2[\\ R(\tau_1) = R(\tau_2) = \dfrac{I}{aS} \end{cases} \\ * \quad \dot{v}\lambda = \dot{\lambda}\,[R - v] \end{cases}$$



Let $\mathcal{U} = \{v, \lambda\}$

THEOREM :

* For any $\lambda_0 \in [\lambda_T, \lambda_R[$, there exists a branch $\mathcal{U} = \mathcal{U}(R_0) \equiv \{v(R_0), \lambda(R_0)\}$ starting at $\{0, \lambda_0\}$, tangent to the corresponding initial velocity.
* Moreover $\exists\, R_{00} \in [\frac{I}{S}, +\infty[$, & $\mathcal{U}(R_0)$ satisfies a Cauchy pb. of the type:

$$\begin{cases} \dot{\mathcal{U}} = \mathcal{F}(\mathcal{U}, R_0) \\ \mathcal{U}(R_{00}) = \{0, \lambda_0\} \end{cases}$$

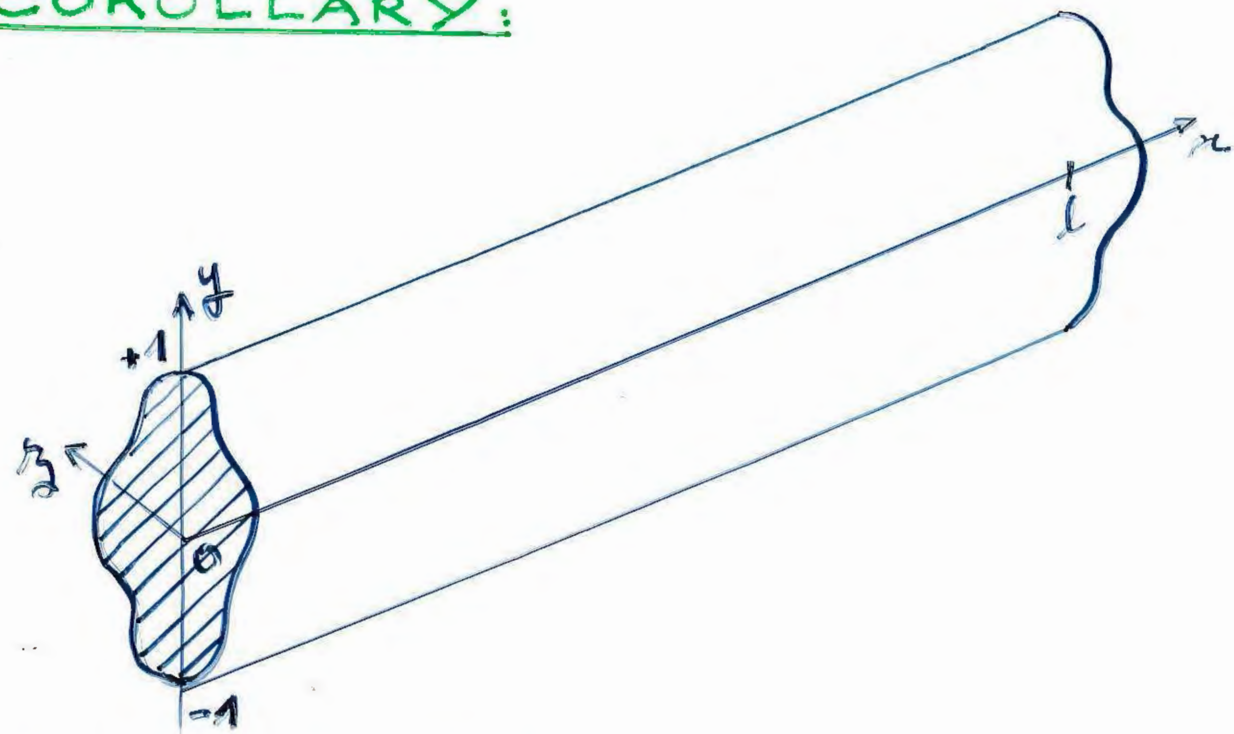
Results from:

$$\begin{cases} \dot{v} = \dot{\lambda}\, \mathcal{N}(\lambda, v, R_0) \\ \dfrac{\partial \Phi}{\partial \lambda} \dot{\lambda} + \dfrac{\partial \Phi}{\partial v} \dot{v} + \dfrac{\partial \Phi}{\partial R_0} \dot{R}_0 = 0 \end{cases}$$

then :

$$\left\{ \frac{\partial \Phi}{\partial \lambda} + \frac{\partial \Phi}{\partial v} \mathcal{N} \right\} \dot{\lambda} + \frac{\partial \Phi}{\partial R_0} = 0$$

COROLLARY:



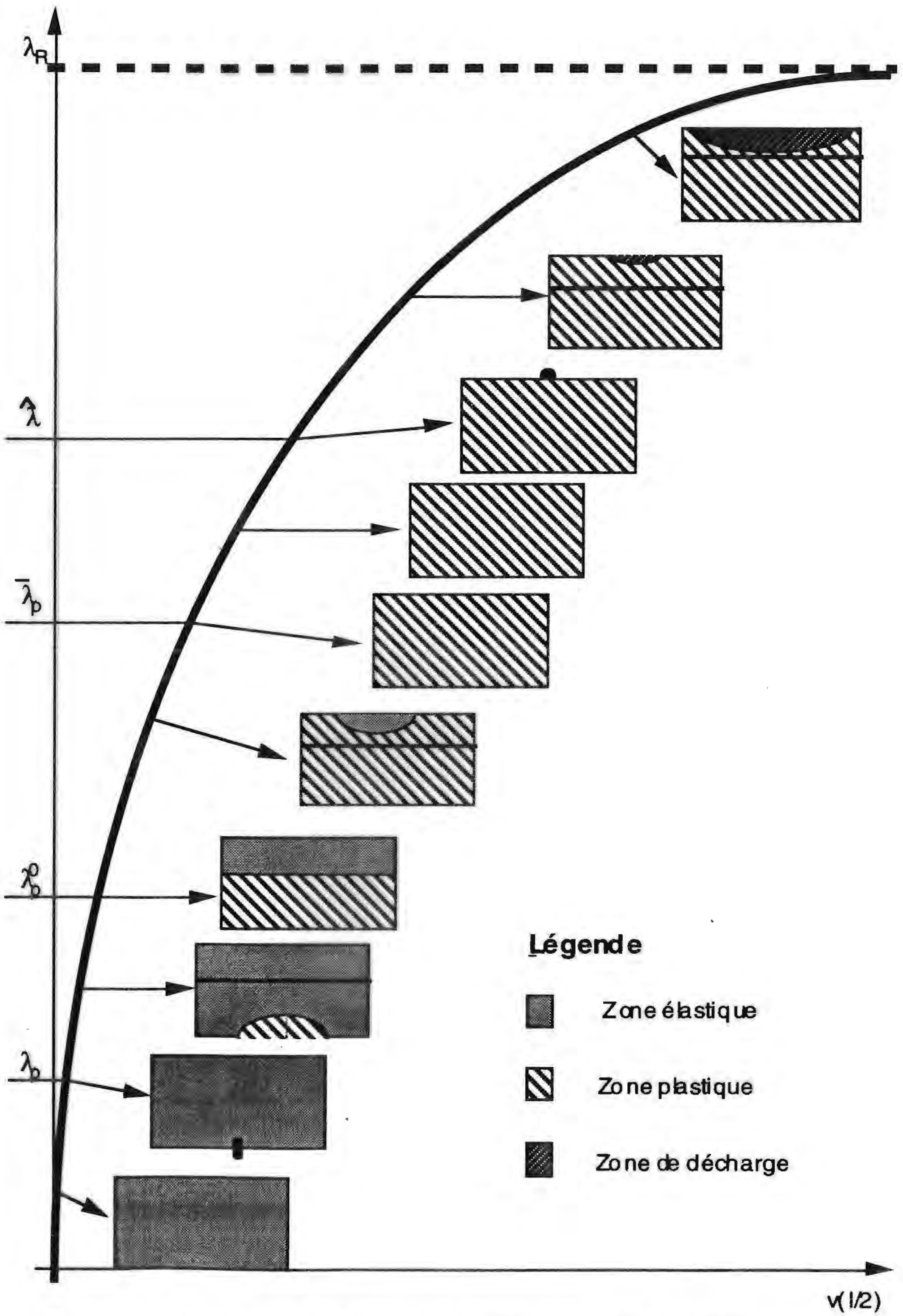
Assume the thickness of the cross. section is a $\mathcal{C}^k$ (resp. analytic) of $[-1,+1]$.

Then :

* The branches $\mathcal{U}(R_0)$ are $\mathcal{C}^{k+1}$ (resp. anal.)
* For any given $x \in]0, l/2[$, and any initial data $\{0, \lambda_0\}$, λ_0 given by the initial velocity problem, the curves $\lambda(R_0)$ and $v(x, R_0)$ are monotonous, strictly increasing, and satisfy:

$$\begin{cases} \lim\limits_{R_0 \to +\infty} \lambda(R_0) = \lambda_R \\ \lim\limits_{R_0 \to +\infty} v(x, R_0) = +\infty \end{cases}$$

By the way, the problem of the maximal loads has been revisited ....



Évolution des frontières libres en fonction du chargement