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Instabilities and local bifurcations
Elements of Theory

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1 Introduction

These lectures are extracted from the book "Local Bifurcations, center Manifolds, and Normal Forms in infinite-dimensional Dynamical systems", EDP Sciences and Springer Universitext 2011, co-authored with Mariana Haragus.

We restrict our attention to the study of local bifurcations. Starting with the simplest bifurcation problems arising for ordinary differential equations in one and two dimensions, the purpose of these notes is to describe several tools from the theory of infinite-dimensional dynamical systems, allowing to treat more complicated bifurcation problems, as for instance bifurcations arising in partial differential equations. Such tools are extensively used to solve concrete problems arising in physics, mechanics and natural sciences.

In a parameter-dependent physical system, for example, modelized by a differential equation, the presence of a bifurcation corresponds to a topological change in the structure of the solution set (which may break its symmetry in the case of a system invariant under some symmetry group). Such a change may imply the occurrence of new solutions, or the disappearance of certain solutions, or may indicate a change of stability of certain solutions. Local bifurcation theory allows one to detect solutions and to describe their geometric (including symmetries) and dynamic properties.

For instance, in the classical Couette–Taylor problem describing flows between two coaxial rotating cylinders (briefly presented in Section 6), the theory was not only a qualitative one, but also sufficiently quantitative to allow prediction of numerical values of the parameters, where new flows, such as "ribbons," were expected to be observed. These were indeed later observed experimentally [67]. This predictive power of the local theory appeared again in water wave theory, or in the propagation of interfaces between metastable states (see [21] chapter 5).

We focus here on two specific methods that arise in the analysis of local bifurcations in infinite-dimensional systems, namely the center manifold reduction and the normal form theory. Center manifolds provide a powerful method of analysis of such systems, as they allow one to reduce, under certain conditions, the infinite-dimensional dynamics near a bifurcation point to a finite-dimensional dynamics, described by a system of ordinary differential equations. An efficient way of studying the resulting reduced systems is with the help of normal form theory, which consists in suitably transforming a nonlinear system, in order to keep only the relevant nonlinear terms and to allow easier recognition of its dynamics. The combination of these two methods led over the recent years to significant progress in the understanding of various problems arising in applied sciences. A common feature of many of these problems is the presence of symmetries. It turns out that both the center manifold reduction and the normal form transformations preserve symmetries, allowing then an efficient treatment of such problems. In addition, they provide a detailed comprehensive study near a singularity in the solution set of the system, which might also orient a numerical treatment of such problems.

In sections 2 and 3 we discuss typical local bifurcations in one and two dimensions. We restrict our attention to bifurcations of codimension 1 which require only one real parameter in order to generically occur. We include several cases of systems that possess an invariance under some simple symmetry. Section 4 is devoted to the center manifold theory. This is the core tool used all throughout these notes. We present the center manifold reduction for infinite-dimensional systems, together with simple examples and exercises illustrating the variety of possible applications. The aim is to allow readers who are not familiar with the subject to use this reduction method simply by checking some clear assumptions. Section 5 is concerned with the normal form theory. In particular, we show how to systematically compute the normal forms in concrete situations. We illustrate the general theory on different bifurcation problems, for which we provide explicit formulas for the normal form, allowing one to obtain quantitative results for the resulting systems.

Finally, in section 6 we present some applications of the methods described in the previous sections. Without going into detail, for which we refer to the literature, we discuss hydrodynamic instabilities arising in the Couette–Taylor and the Bénard–Rayleigh convection problems.

Historical Remark. *Many authors refer to the work of C. G. J. Jacobi from 1834, on equilibria of self- gravitating rotating ellipsoids [34], as a first reference in the field of bifurcation theory. However, it seems that the first serious works on bifurcation problems were by Archimedes and Apollonios over 200 years BCE. Archimedes studied the equilibria of a floating paraboloid of revolution [61]. In today's terminology his results would correspond to a pitchfork bifurcation which breaks a flip symmetry, or to a steady bifurcation with* $O(2)$ *symmetry, when taking into account the invariance under rotations about the paraboloid axis. Apollonios studied the extrema of the length of segments joining a point of the plane to a given conic [37]. The number of solutions changes from one to three in crossing the envelope of the normals to the conic. Here again, due to the symmetry of the conic, we have an example of a pitchfork bifurcation. Finally, it seems that the French word "bifurcation" was introduced by Poincaré in 1885 [58].*

2 Bifurcations in Dimension 1

We consider in this section two generic bifurcations that are found for scalar differential equations of the form

$$\frac{du}{dt} = f(u, \mu). \tag{2.1}$$

Here the unknown u is a real-valued function of the "time" t, and the vector field f is real-valued depending, besides u, upon a real parameter μ. The parameter μ is the *bifurcation parameter*.

We assume that the vector field f is of class C^k, $k \geq 2$, in a neighborhood of $(0,0)$ satisfying

$$f(0,0) = 0, \quad \frac{\partial f}{\partial u}(0,0) = 0. \tag{2.2}$$

The first condition shows that $u = 0$ is an equilibrium of (2.1) at $\mu = 0$. We are interested in (local) bifurcations that occur in the neighborhood of this equilibrium when we vary the parameter μ. Then the second equality in (2.2) is a necessary, but not sufficient, condition for the appearance of local bifurcations at $\mu = 0$. If $\partial f/\partial u(0,0) \neq 0$, the condition (2.2) is not satisfied and a direct application of the implicit function theorem shows that the equation $f(u,\mu) = 0$ possesses a unique solution $u = u(\mu)$ in a neighborhood of 0, for any μ sufficiently small. In particular, $u = 0$ is the only equilibrium of (2.1) in a neighborhood of the origin when $\mu = 0$, and the same property holds for μ sufficiently small. Furthermore, it is not difficult to show that the dynamics of (2.1) in a neighborhood of the origin is qualitatively the same for all sufficiently small values of the parameter μ. Consequently, in this situation no bifurcation occurs for small values of μ.

2.1 Saddle-Node Bifurcation

We discuss in this section the simplest bifurcation that occurs in one dimension, the *saddle-node bifurcation*. Throughout this section we make the following hypothesis.

Hypothesis 2.1. *Assume that the vector field f is of class C^k, $k \geq 2$, in a neighborhood of $(0,0)$, and that it satisfies (2.2) and*

$$\frac{\partial f}{\partial \mu}(0,0) =: a \neq 0, \quad \frac{\partial^2 f}{\partial u^2}(0,0) =: 2b \neq 0. \tag{2.3}$$

An immediate consequence of this hypothesis is that f has the expansion

$$f(u,\mu) = a\mu + bu^2 + o(|\mu| + u^2),$$

as $(u,\mu) \to (0,0)$. It is then natural to start by studying the truncated equation

$$\frac{du}{dt} = a\mu + bu^2, \tag{2.4}$$

for which we expect that the dynamics near 0 are the same as those of (2.1).

Truncated Equation

The equilibria of (2.4) are solutions of the equation $a\mu + bu^2 = 0$, so that the truncated equation has no equilibria if $ab\mu > 0$, one equilibrium $u = 0$ if $\mu = 0$, and a pair of equilibria $u = \pm\sqrt{-a\mu/b}$ if $ab\mu < 0$. As for the dynamics, in the case $ab\mu > 0$ the function $a\mu + bu^2$ has a constant sign for all $u \in \mathbb{R}$, so that the solutions are monotone: increasing when $b > 0$ and decreasing when $b < 0$ (see Figure 2.1(a)). The same property holds for the nonequilibrium

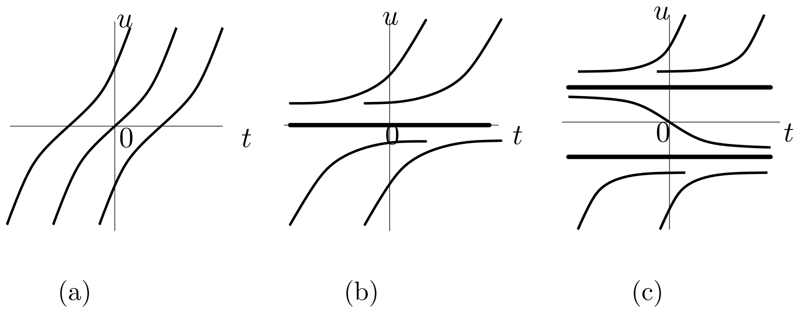


Figure 2.1: Extended phase portrait, in the (t, u)-plane, of the truncated equation (2.4) for $b > 0$ and (a) $a\mu > 0$, (b) $\mu = 0$, (c) $a\mu < 0$.

solutions in the case $\mu = 0$: They are increasing when $b > 0$ and decreasing when $b < 0$ (see Figure 2.1(b)). Finally, in the case $ab\mu < 0$, the function $a\mu + bu^2$ changes sign at the equilibrium points $u = \pm\sqrt{-a\mu/b}$, and we find that solutions with $|u(t)| < \sqrt{-a\mu/b}$ are decreasing when $b > 0$ and increasing when $b < 0$, whereas solutions with $|u(t)| > \sqrt{-a\mu/b}$ are increasing when $b > 0$ and decreasing when $b < 0$ (see Figure 2.1(c)). In particular, the equilibrium $-\sqrt{-a\mu/b}$ is attractive, asymptotically stable, when $b > 0$, and repelling, unstable, when $b < 0$; whereas, the equilibrium $\sqrt{-a\mu/b}$ has opposite stability properties.

We summarize in Figure 2.2 the dynamics of the truncated equation. In all cases, the qualitative behavior of the solutions changes when μ crosses 0. The value $\mu = 0$ is the bifurcation point. At this value, a pair of equilibria with opposite stability properties emerges for $\mu > 0$ when $ab < 0$, and $\mu < 0$ when $ab > 0$. We are here in the presence of a *saddle-node bifurcation* (also called *fold* or *turning point* bifurcation).

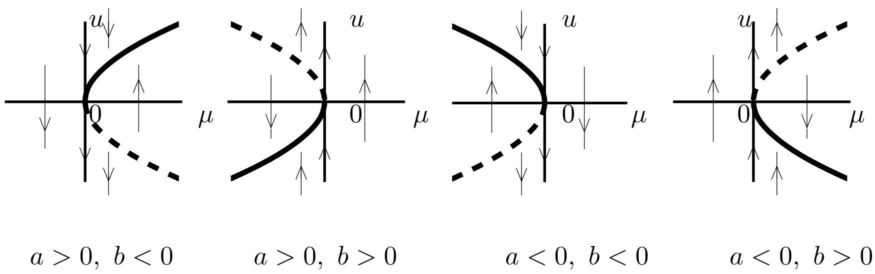


Figure 2.2: Saddle-node bifurcation: bifurcation diagrams, in the (μ, u)-plane, of the truncated equation (2.4) for different values of a and b. The solid lines represent branches of stable equilibria, the dashed lines branches of unstable equilibria, and the arrows indicate the sense of increasing time t. For the full equation (2.1), under Hypothesis 2.1, the bifurcation diagrams are qualitatively the same in a neighborhood of the origin.

Remark 2.2 (Saddle-node bifurcation)**.** *The names* fold *and* turning point bifurcations *are inspired by the form of the branch of the bifurcating equilibria in the (μ, u)-plane. The name* saddle-node bifurcation *comes from the fact that in the n-dimensional case, when $u(t) \in \mathbb{R}^n$, the two emerging equilibria are typically a saddle point and a node.*

Remark 2.3 (Explicit solutions)**.** *The truncated equation (2.4) can be easily solved explic-*

itly. For $ab\mu > 0$ we set

$$u = \sqrt{\frac{a\mu}{b}}v$$

and obtain the equation

$$\frac{dv}{dt} = \operatorname{sign}(b)\sqrt{ab\mu}\,(1 + v^2).$$

The unique solution v of this first order ordinary differential equation (ODE) with initial data $v(0) = v_0$ is then given by

$$v(t) = \tan\Big(\operatorname{sign}(b)\sqrt{ab\mu}\,t + \arctan(v_0)\Big)\,, \tag{2.5}$$

with $\arctan v_0 \in (-\pi/2, \pi/2)$. *Similarly, for $ab\mu < 0$ we set*

$$u = \sqrt{\frac{-a\mu}{b}}v$$

and obtain

$$\frac{dv}{dt} = \operatorname{sign}(b)\sqrt{-ab\mu}\,(v^2 - 1).$$

Hence,

$$\frac{v(t)+1}{v(t)-1} = \frac{v_0+1}{v_0-1}e^{-2\operatorname{sign}(b)\sqrt{-ab\mu}\,t},$$

for any $v_0 \neq \pm 1$. For $v_0 = \pm 1$, we find the constant solutions $v(t) = \pm 1$. Finally, for $\mu = 0$ we have the unique solution

$$u(t) = \frac{u_0}{1 - bu_0 t}$$

for initial data $u(0) = u_0$. These calculations then give the results described above and summarized in Figures 2.1 and 2.2. In addition, they show that the solutions blow up in finite time (either positive or negative), except for initial data $u_0 \in \Big[-\sqrt{-a\mu/b}, \sqrt{-a\mu/b}\Big]$, when $ab\mu \leq 0$.

Full Dynamics

Let us now consider the full equation (2.1). The equilibria are solutions of the equation $f(u,\mu) = 0$. Since $a \neq 0$ we can apply the implicit function theorem, which shows that this equation possesses a unique solution $\mu = g(u)$ for u close to 0. The map g is of class $\mathcal{C}^k$ in a neighborhood of the origin, and $g(0) = 0$. Moreover, its Taylor expansion is given by

$$\mu = -\frac{b}{a}u^2 + o(u^2).$$

This gives a curve in the (μ, u)-plane, which has a second order tangency at $(0,0)$ to the parabola $\mu = -bu^2/a$ found for the truncated equation (see Figure 2.2). In particular, this shows that the truncated equation and the full equation have the same number of equilibria in a neighborhood of the origin, which are $o(|\mu|^{1/2})$-close to each other. Consequently, the full equation has no equilibria if $ab\mu > 0$, one equilibrium $u = 0$ if $\mu = 0$, and a pair of equilibria $u_\pm(\mu) = \pm\sqrt{-a\mu/b} + o(|\mu|^{1/2})$ if $ab\mu < 0$.

As for the dynamics, the situation is also similar to that for the truncated equation, provided u and μ are sufficiently small. In the case $ab\mu > 0$ the function $f(u, \mu)$ has constant sign for sufficiently small u and μ, so that in a neighborhood of the origin the solutions are monotone: increasing when $b > 0$ and decreasing when $b < 0$ (see Figure 2.1(a)). When $\mu = 0$, the nonequilibrium solutions are monotone: increasing when $b > 0$ and decreasing when $b < 0$ (see Figure 2.1(b)). Finally, in the case $ab\mu < 0$, the function $f(u, \mu)$ changes sign at the equilibrium points $u_\pm(\mu)$, where

$$\frac{\partial f}{\partial u}(u_\pm(\mu), \mu) = 2bu_\pm(\mu) + o(|\mu|^{1/2})$$

has a definite sign. Then the equilibrium $u_-(\mu)$ is attractive, asymptotically stable when $b > 0$, and repelling, unstable when $b < 0$; whereas, the equilibrium $u_+(\mu)$ has opposite stability properties. Further, we find that solutions with $u(t) \in (u_-(\mu), u_+(\mu))$ are decreasing when $b > 0$ and increasing when $b < 0$, whereas solutions outside this interval, with $u(t) > u_+(\mu)$ or $u(t) < u_-(\mu)$ are increasing when $b > 0$ and decreasing when $b < 0$ (see Figure 2.1(c)). Just as for the truncated equation, we have here a *saddle-node bifurcation* (see Figure 2.2). We summarize this result in the following theorem.

Theorem 2.4 (Saddle-node bifurcation)**.** *Assume that the vector field f satisfies Hypothesis 2.1. Then, for the differential equation (2.1) a saddle-node bifurcation occurs at $\mu = 0$. More precisely, the following properties hold in a neighborhood of 0 in $\mathbb{R}$ for sufficiently small μ:*

(i) If $ab < 0$ (resp., $ab > 0$) the differential equation has no equilibria for $\mu < 0$ (resp., for $\mu > 0$).

(ii) If $ab < 0$ (resp., $ab > 0$), the differential equation possesses precisely two equilibria $u_\pm(\varepsilon)$, $\varepsilon = |\mu|^{1/2}$ for $\mu > 0$ (resp., for $\mu < 0$), with opposite stabilities. Furthermore, the map $\varepsilon \mapsto u_\pm(\varepsilon)$ is of class $\mathcal{C}^{k-2}$ in a neighborhood of 0, and $u_\pm(\varepsilon) = O(\varepsilon)$.

Remark 2.5 (Higher orders)**.** *In the case when $b = 0$, but still $a \neq 0$, one has to look for the lowest positive integer n for which the derivative $\partial^n f/\partial u^n(0, 0) = bn! \neq 0$. The equilibria are then of order $O(|\mu|^{1/n})$, and for n even the qualitative phase portraits are as in Figure 2.2. When n is odd, the branch of equilibria crosses the u-axis, and on each side the equilibria have the same stability (stable if $b < 0$, or unstable if $b > 0$). If $a = b = 0$, then the situation requires a study of the Newton polygon and enters more into the framework of singularity theory (e.g., see [17]).*

2.2 Pitchfork Bifurcation

In many physical situations the problem possesses some symmetry. The simplest one that occurs in one dimension is the *reflection*, or *mirror, symmetry*: $u \mapsto -u$. In this section we discuss this situation and the corresponding generic bifurcation, which is the *pitchfork bifurcation*.

We consider again the scalar differential equation (2.1) and now make the following assumptions.

Hypothesis 2.6. *Assume that the vector field in (2.1) is of class* $\mathcal{C}^k$*,* $k \geq 3$*, in a neighborhood of* $(0,0)$*, that it satisfies (2.2), and that it is odd with respect to* u*, i.e.,*

$$f(-u,\mu) = -f(u,\mu). \tag{2.6}$$

Further assume that

$$\frac{\partial^2 f}{\partial\mu\partial u}(0,0) =: a \neq 0, \quad \frac{\partial^3 f}{\partial u^3}(0,0) =: 6b \neq 0. \tag{2.7}$$

An immediate consequence of the oddness property of f is that

$$f(0,\mu) = 0 \text{ for all } \mu,$$

so that $u = 0$ is an equilibrium of (2.1) for all μ.

Truncated Equation

We start again by studying the truncated equation, which in this case is

$$\frac{du}{dt} = a\mu u + bu^3. \tag{2.8}$$

As for the full equation, $u = 0$ is an equilibrium of this equation for all values of μ. Upon solving the equation $a\mu u + bu^3 = 0$, we find that $u = 0$ is the only equilibrium of (2.8) if $ab\mu \geq 0$, and that for $ab\mu < 0$ there is an additional pair of nontrivial equilibria $u = \pm\sqrt{-a\mu/b}$. As for the dynamics, the nonequilibrium solutions are monotone, with monotonicity determined by the sign of the function $a\mu u + bu^3$. This function changes sign precisely at the equilibrium points, and a direct calculation leads to the diagram in Figure 2.3.

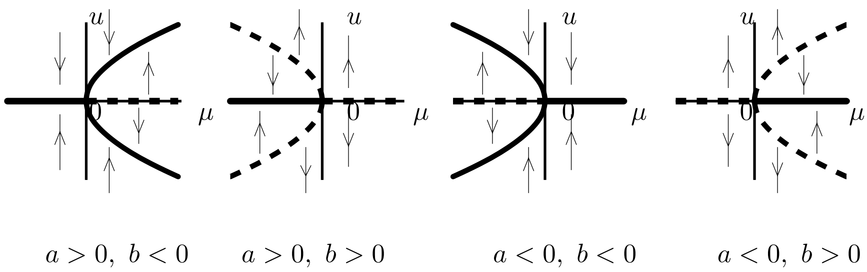


Figure 2.3: Pitchfork bifurcation: bifurcation diagrams, in the (μ,u)-plane, of the truncated equation (2.8) for different values of a and b. The solid lines represent branches of stable equilibria, the dashed lines branches of unstable equilibria, and the arrows indicate the sense of increasing time t. For the full equation (2.1), under Hypothesis 2.6, the bifurcation diagrams are qualitatively the same.

Again, the qualitative behavior of the solutions changes when μ crosses 0, so that $\mu = 0$ is a bifurcation point. At this value, the trivial equilibrium $u = 0$ changes its stability, and a pair of equilibria having the same stability, but opposite to that of the trivial equilibrium, emerges for $\mu > 0$ when $ab < 0$, and $\mu < 0$ when $ab > 0$. Here we are in the presence of a *pitchfork bifurcation*. The cases in which the emerging nontrivial equilibria are stable are called *supercritical*, whereas the cases in which these equilibria are unstable are called *subcritical*.

Remark 2.7 (Pitchfork bifurcation)**.** *The name* pitchfork bifurcation *comes from the form of the branches of equilibria in the bifurcation diagram (even though actual pitchforks in the countryside may look different in various countries).*

Remark 2.8 (Explicit solution)**.** *The truncated equation (2.8) can be easily solved explicitly. A direct calculation shows that its unique solution for initial data $u(0) = u_0$ is given by*

$$u^2(t) = \frac{a\mu u_0^2}{a\mu e^{-2a\mu t} + bu_0^2(e^{-2a\mu t} - 1)}.$$

This formula allows us to recover the bifurcation diagrams in Figure 2.3 and shows, in addition, that the unbounded nonequilibrium solutions blow up in either positive or negative finite time.

Full Dynamics

We consider now the full equation (2.1), under Hypothesis 2.6. The equilibria are solutions of $f(u, \mu) = 0$, and as already noticed, $u = 0$ is always an equilibrium because of the oddness of f in u. In addition, a standard analysis argument shows that we can rewrite the vector field f as follows:

$$f(u, \mu) = uh(u^2, \mu), \quad h(u^2, \mu) = a\mu + bu^2 + o(|\mu| + u^2),$$

where h is of class $\mathcal{C}^{(k-1)/2}$ in a neighborhood of $(0, 0)$. Since $a \neq 0$ we can apply the implicit function theorem to the equation $h(u^2, \mu) = 0$, which shows that it has a unique solution $\mu = g(u^2)$ with $g(0) = 0$ and g of class $\mathcal{C}^{(k-1)/2}$ in a neighborhood of 0. The Taylor expansion of g is given by

$$\mu = -\frac{b}{a}u^2 + o(u^4).$$

We then conclude that there is a curve of nontrivial equilibria in the (μ, u)-plane that has a second order tangency at $(0, 0)$ to the parabola $\mu = -bu^2/a$ found for the truncated equation (see Figure 2.3), and which is symmetric with respect to the μ-axis. Again, this shows that the truncated equation and the full equation have the same number of equilibria in a neighborhood of the origin, which are $o(|\mu|^{1/2})$-close to each other. As for the dynamics, it is here again easy to study by looking at the sign of $f(u, \mu)$. The arguments are analogous to those in the case discussed in Section 2.1 and lead to the bifurcation diagrams in Figure 2.3. We summarize these results in the following theorem.

Theorem 2.9 (Pitchfork bifurcation)**.** *Assume that the vector field f satisfies Hypothesis 2.6. Then, for the differential equation (2.1), a supercritical (resp., subcritical) pitchfork bifurcation occurs at $\mu = 0$ when $b < 0$ (resp., $b > 0$). More precisely, the following properties hold in a neighborhood of* 0 *in* $\mathbb{R}$ *for sufficiently small μ:*

(i) If $ab < 0$ (resp., $ab > 0$) the differential equation has precisely one trivial equilibrium $u = 0$ for $\mu < 0$ (resp., for $\mu > 0$). This equilibrium is stable when $b < 0$ and unstable when $b > 0$.

(ii) If $ab < 0$ (resp., $ab > 0$), the differential equation possesses, for $\mu > 0$ (resp., for $\mu < 0$), the trivial equilibrium $u = 0$ and two nontrivial equilibria $u_\pm(\varepsilon)$, $\varepsilon = |\mu|^{1/2}$, which are symmetric, $u_-(\varepsilon) = -u_+(\varepsilon)$. The map $\varepsilon \mapsto u_\pm(\varepsilon)$ is of class $\mathcal{C}^{k-3}$ in a neighborhood of 0, and $u_\pm(\varepsilon) = O(\varepsilon)$. Furthermore, the nontrivial equilibria are stable when $b < 0$ and unstable when $b > 0$, whereas the trivial equilibrium has opposite stability.

Remark 2.10 (Higher orders). *In the case $b = 0$, but still $a \neq 0$, one has to look for the lowest n for which the derivative $\partial^{2n+1} f/\partial u^{2n+1}(0,0) \neq 0$. The equilibria are then of order $O(|\mu|^{1/2n})$ and the qualitative phase portraits are as in Figure 2.3. If $a = b = 0$ then the situation requires a study of the Newton polygon and belongs more in the field of singularity theory (e.g., see [17]).*

3 Bifurcations in Dimension 2

In the remainder of this chapter we consider differential equations in $\mathbb{R}^2$,

$$\frac{du}{dt} = \mathbf{F}(u, \mu). \tag{3.1}$$

Now the unknown u takes values in $\mathbb{R}^2$, just as the vector field $\mathbf{F}$, which depends again besides depending on u, upon a real parameter μ.

We assume that the vector field $\mathbf{F}$ is of class $\mathcal{C}^k$, $k \geq 3$, in a neighborhood of $(0,0)$, satisfying

$$\mathbf{F}(0,0) = 0. \tag{3.2}$$

Again, this condition shows that $u = 0$ is an equilibrium of (3.1) at $\mu = 0$. We are interested in (local) bifurcations which occur in the neighborhood of this equilibrium when varying the parameter μ. The appearance, or the absence, of bifurcations is in this case determined by the linearization of the vector field at $(0,0)$,

$$\mathbf{L} := D_u\mathbf{F}(0,0),$$

which is a linear map (operator) acting in $\mathbb{R}^2$. In the case when the linear map $\mathbf{L}$ has no eigenvalue on the imaginary axis, the Hartman–Grobman theorem shows that the phase portraits of the equation (3.1) are qualitatively the same upon varying μ in a neighborhood of 0 (e.g., see [19], [22]). In particular, no local bifurcations occur in this case. When $\mathbf{L}$ has eigenvalues on the imaginary axis, bifurcations may occur at $\mu = 0$. The type of these bifurcations depend upon the location of the eigenvalues on the imaginary axis. While we do not attempt to give a complete description of the possible bifurcations for two-dimensional systems, we focus in this section on two cases: $\mathbf{L}$ has a pair of complex conjugated purely imaginary eigenvalues (Hopf bifurcation), and $\mathbf{L}$ has a double zero eigenvalue (steady bifurcation) for a system possessing an $O(2)$-symmetry. The cases in which 0 is a simple eigenvalue of **L** and another eigenvalue is real and different from 0, fall in the discussion of Section 4, using a center manifold reduction (e.g., see the example in Subsection 4.4.1). The case of 0 a double, non-semisimple (with only one eigenvector) eigenvalue of **L** is treated in Section 5.

3.1 Hopf Bifurcation

One generic bifurcation in two dimensions is the *Hopf bifurcation*, which occurs when the linear operator **L** possesses a pair of purely imaginary complex conjugated eigenvalues. This bifurcation was first proved in two dimensions by Andronov [1] in 1937; it is therefore also referred to as Andronov–Hopf bifurcation, after it was guessed by H. Poincaré in the early 1900s [57]. The n-dimensional case was proved by Hopf in 1942, using the Lyapunov–Schmidt method [24]. Our analysis relies upon the *normal form* theory that we develop in detail in section 5.

Hypothesis 3.1. *Assume that the vector field* $\mathbf{F}$ *in (3.1) is of class* $\mathcal{C}^k$, $k \geq 5$, *that it satisfies (3.2), and that the two eigenvalues of the linear operator* $\mathbf{L}$ *are* $\pm i\omega$ *for some* $\omega > 0$.

Remark 3.2. *(i) Since the operator* $\mathbf{L}$ *is real, its spectrum is symmetric with respect to the real axis, so that purely imaginary eigenvalues occur in pairs* $\pm i\omega$.

(ii) Hypothesis 3.1 implies that $\mathbf{L}$ *is invertible, since* 0 *is not an eigenvalue of* $\mathbf{L}$. *By arguing using the implicit function theorem, we can then solve the equation* $\mathbf{F}(u,\mu) = 0$ *near* $(0,0)$. *This gives a unique family of steady solutions* $u = u(\mu)$ *for sufficiently small* μ, *with* $u(0) = 0$. *Furthermore, the map* $\mu \mapsto u(\mu)$ *is of class* $\mathcal{C}^k$, *and by making the change of variables* $u \mapsto u(\mu) + u$, *we may replace assumption (3.2) by*

$$\mathbf{F}(0,\mu) = 0. \tag{3.3}$$

In this way, $u = 0$ *becomes an equilibrium of (3.1) for all values of* μ *sufficiently small. We point out that it is then a generic possibility that a pair of complex eigenvalues of the linearized operators* $\mathbf{L}_\mu = D_u\mathbf{F}(0,\mu)$ *crosses the imaginary axis for a critical value of the parameter* μ *(here* $\mu = 0$*).*

(iii) In contrast to the two bifurcations discussed before, now the number of equilibria of the differential equation stays constant upon varying μ *in a neighborhood of* 0. *As we shall see, we have here a different type of bifurcation in which it is the dynamics of the differential equation that change at the bifurcation point* $\mu = 0$, *and not the number of equilibria. Such bifurcations are also called* dynamic bifurcations, *whereas those in which the number of equilibria changes are also called* steady bifurcations.

Consider the eigenvectors ζ and $\overline{\zeta}$ associated with the eigenvalues $i\omega$ and $-i\omega$ of $\mathbf{L}$, respectively,

$$\mathbf{L}\zeta = i\omega\zeta, \quad \mathbf{L}\overline{\zeta} = -i\omega\overline{\zeta}.$$

A convenient way of looking at equation (3.1) in this case is by representing any $u \in \mathbb{R}^2$ by a complex coordinate $z \in \mathbb{C}$ through

$$u = z\zeta + \overline{z}\overline{\zeta}. \tag{3.4}$$

Adopting the same decomposition for $\mathbf{F}$, we write

$$\mathbf{F}(u,\mu) = f(z,\overline{z},\mu)\zeta + \overline{f}(z,\overline{z},\mu)\overline{\zeta}$$

and then obtain two complex differential equations

$$\frac{dz}{dt} = f(z,\overline{z},\mu), \tag{3.5}$$

together with its complex conjugate. The complex-valued vector field f is of class $\mathcal{C}^k$ in a neighborhood of the origin in $\mathbb{R}^2 \times \mathbb{R}$, where the argument in $\mathbb{R}^2$ is represented by the "diagonal" $(z,\overline{z}) \in \mathbb{C}^2$. (Notice that f is not holomorphic in z.) In these coordinates, the differential of the new vector field $(f,\overline{f})$ at the origin is given by

$$\mathbf{L} = \begin{pmatrix} \frac{\partial f}{\partial z}(0,0,0) & \frac{\partial f}{\partial \overline{z}}(0,0,0) \\ \frac{\partial \overline{f}}{\partial z}(0,0,0) & \frac{\partial \overline{f}}{\partial \overline{z}}(0,0,0) \end{pmatrix} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}.$$

Though the linear part $\mathbf{L}$ of (3.5) is now in canonical form, it is still difficult to detect its dynamics in general. Our approach relies upon the normal form theory developed in section 5. Roughly speaking, the idea of normal forms consists in adding a polynomial term to the change of coordinates (3.4), such that the vector field of the resulting system has a simpler, particular form, also at the nonlinear level.

Normal Form

According to the general normal form theorem, Theorem 5.21, for any integer $p \le k$, and any μ sufficiently small, there exists a polynomial $\mathbf{\Phi}_\mu$ of degree p in $(A,\overline{A})$, with complex coefficients, depending upon μ, and taking values in $\mathbb{R}^2$, such that

$$\mathbf{\Phi}_0(0,0) = 0, \quad \partial_A \mathbf{\Phi}_0(0,0) = 0, \quad \partial_{\overline{A}} \mathbf{\Phi}_0(0,0) = 0,$$

and that the (near to identity) change of variables in $\mathbb{R}^2$,

$$u = A\zeta + \overline{A\zeta} + \mathbf{\Phi}_\mu(A,\overline{A}), \quad A \in \mathbb{C}, \tag{3.6}$$

transforms the equation (3.1) into a differential equation, or "amplitude equation,"

$$\frac{dA}{dt} = i\omega A + N_\mu(A,\overline{A},) + \rho(A,\overline{A},\mu). \tag{3.7}$$

Here N_μ is a complex polynomial of degree p in $(A,\overline{A})$, with

$$N_0(0,0) = 0, \quad \partial_A N_0(0,0) = 0, \quad \partial_{\overline{A}} N_0(0,0) = 0, \tag{3.8}$$

and the remainder ρ satisfies

$$\rho(A,\overline{A},\mu) = o(|A|^p).$$

Furthermore, the polynomial

$$\mathbf{N}_\mu(A,\overline{A}) = (N_\mu(A,\overline{A}), \overline{N_\mu}(A,\overline{A}))$$

commutes with the mapping

$$(A, \overline{A}) \mapsto (e^{i\omega t} A, e^{-i\omega t} \overline{A}),$$

which implies that

$$N_\mu(e^{i\omega t} A, e^{-i\omega t} \overline{A}) = e^{i\omega t} N_\mu(A, \overline{A}) \text{ for all } A, t. \tag{3.9}$$

Remark 3.3 (Symmetry). *We observe that the transformation (3.6) has the effect of* adding a symmetry for the terms up to degree p *in the expansion of the transformed vector field. The property (3.9) means that the truncation at order p of the vector field is equivariant under rotations in the complex plane, which is a rather strong restriction. We point out that in general this transformation cannot be achieved for $p = \infty$, even when $\mathbf{F}$ in (3.1) is analytic.*

The following elementary lemma, proved in [21], allows us to describe more precisely the polynomials N_μ satisfying (3.9).

Lemma 3.4. *Let f be a complex-valued function of class $\mathcal{C}^k$, $k \geq 1$, defined in a neighborhood $\mathcal{U}$ of the origin in $\{(z, \overline{z})\ ;\ z \in \mathbb{C}\}$, and which verifies*

$$f(e^{i\omega t} z, e^{-i\omega t} \overline{z}) = e^{i\omega t} f(z, \overline{z}) \text{ for any } t \in \mathbb{R} \text{ and } (z, \overline{z}) \in \mathcal{U}. \tag{3.10}$$

Then there exists an even, complex-valued function g of class $\mathcal{C}^{k-1}$ defined in a neighborhood of 0 in $\mathbb{R}$, such that

$$f(z, \overline{z}) = z g(|z|). \tag{3.11}$$

Furthermore, if f is a polynomial, then g is an even polynomial, $g(|z|) = \phi(|z|^2)$, for a polynomial ϕ.

Going back to the differential equation (3.7), the above lemma together with the equalities (3.8) show that it is of the form

$$\frac{dA}{dt} = i\omega A + A Q(|A|^2, \mu) + \rho(A, \overline{A}, \mu). \tag{3.12}$$

Here Q is a complex-valued polynomial with expansion

$$Q(|A|^2, \mu) = a\mu + b|A|^2 + O((|\mu| + |A|^2)^2), \tag{3.13}$$

in which a and b are complex numbers. We make the following generic assumption on the coefficients a and b.

Hypothesis 3.5. *The complex coefficients a and b in the expansion (3.13) of the polynomial Q have nonzero real parts, $a_r \neq 0$ and $b_r \neq 0$.*

Truncated System

We start again by the study of the truncated system obtained by suppressing the higher order terms ρ in (3.12). We introduce polar coordinates by setting

$$A = re^{i\phi},$$

where $r \in \mathbb{R}^+$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. We obtain the equation

$$\frac{dr}{dt} + ir\frac{d\phi}{dt} = i\omega r + rQ(r^2, \mu),$$

and by taking the real and imaginary parts, we find the system

$$\frac{dr}{dt} = rQ_r(r^2, \mu), \tag{3.14}$$

$$\frac{d\phi}{dt} = \omega + Q_i(r^2, \mu), \tag{3.15}$$

where $Q_r = (Q + \overline{Q})/2$ and $Q_i = (Q - \overline{Q})/2i$ are the real and imaginary parts of the polynomial Q, respectively. Then Q_r and Q_i are polynomials of degree $\leq (p-1)/2$ in r^2, with $Q_r(0,0) = Q_i(0,0) = 0$, and expansions

$$\begin{aligned} Q_r(r^2, \mu) &= a_r\mu + b_r r^2 + O((|\mu| + r^2)^2), \\ Q_i(r^2, \mu) &= a_i\mu + b_i r^2 + O((|\mu| + r^2)^2). \end{aligned}$$

The real coefficients a_r and b_r represent the real parts of a and b, respectively, which are both nonzero, by Hypothesis 3.5, whereas a_i and b_i represent the imaginary parts of a and b, respectively.

The key property of the system (3.14)–(3.15) for r and ϕ is that the *radial equation (3.14) for* r decouples, so that we can solve it separately. Upon comparing (3.14) with the scalar differential equation discussed in Section 2.2, we conclude that for this equation a *pitchfork bifurcation* occurs at $\mu = 0$, which is supercritical when $b_r < 0$ and subcritical when $b_r > 0$. The bifurcation diagrams for this equation are the same as those in Figure 2.3 with a and b replaced by a_r and b_r, respectively. Since for the radial equation we restrict ourselves to positive solutions, then for $a_r b_r < 0$ (resp., $a_r b_r > 0$), the radial equation possesses the positive steady solution

$$r^*(\mu) = \sqrt{-\frac{a_r\mu}{b_r}} + O(|\mu|^{3/2}),$$

for $\mu > 0$ (resp., $\mu < 0$). Upon substituting this solution in the equation (3.15) we obtain the derivative of the phase (pulsation),

$$\frac{d\phi^*(\mu)}{dt} = \omega^*(\mu) = \omega + Q_i((r^*(\mu))^2, \mu) = \omega + \left(a_i - b_i\frac{a_r}{b_r}\right)\mu + O(|\mu|^2),$$

and going back to the amplitude A this gives the periodic solutions

$$A^*(t, \mu) = r^*(\mu)e^{i\omega^*(\mu)t}, \quad t \in \mathbb{R}. \tag{3.16}$$

The stability of these periodic solutions is the same as that of the steady solution $r^*(\mu)$ of (3.14): They are stable when $b_r < 0$ and unstable when $b_r > 0$. Figure 3.1 illustrates the bifurcation diagram in the supercritical case $a_r > 0$ and $b_r < 0$. Similar bifurcation diagrams can be easily obtained in the other three cases, just as for the pitchfork bifurcation in Figure 2.3.

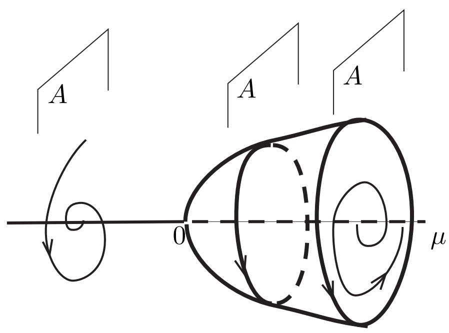


Figure 3.1: Hopf bifurcation in the case $a_r > 0$, $b_r < 0$.

Persistence of Periodic Solutions

We now turn back to the hardest part of the analysis, that is, the proof of the persistence of such periodic orbits for the full equation (3.12). In what follows, we assume $a_r > 0$, $b_r < 0$, and then also $\mu > 0$, to fix ideas, the proof being analogous in the other cases.

As for the truncated system we introduce polar coordinates by setting

$$A = re^{i\phi}, \quad r \in \mathbb{R}^+, \quad \phi \in \mathbb{R}/2\pi\mathbb{Z},$$

and obtain the system

$$\begin{aligned} \frac{dr}{dt} &= f_r(r, \phi, \mu) = rQ_r(r^2, \mu) + R_r(r, \phi, \mu), \\ \frac{d\phi}{dt} &= f_\phi(r, \phi, \mu) = \omega + Q_i(r^2, \mu) + R_\phi(r, \phi, \mu), \end{aligned}$$

where $R_r = O((r + |\mu|)^{p+1})$ and $R_\phi = O((r + |\mu|)^{p+1}/r)$. We now set

$$r = \mu^{1/2} \left(\sqrt{-\frac{a_r}{b_r}} + v \right),$$

where the new unknown v is supposed to lie in a small interval near 0. In this annular region of the plane, for μ small enough,

$$f_\phi(r, \phi, \mu) = \omega + O(\mu)$$

has a constant sign, and

$$f_r(r, \phi, \mu) = \mu^{3/2} \left(\sqrt{-\frac{a_r}{b_r}} + v \right) \left(2vb_r \sqrt{-\frac{a_r}{b_r}} + b_r v^2 \right) + O(\mu^{(p+1)/2}).$$

Using the fact that we can choose $p \geq 4$, this leads to the equation

$$\frac{dv}{d\phi} = -\frac{2a_r\mu}{\omega}v + \rho_1(v,\phi,\mu), \quad \rho_1(v,\phi,\mu) = O(\mu v^2 + \mu^2), \tag{3.17}$$

where ρ_1 is Lipschitz-continuous and bounded for $-\varepsilon < v < \varepsilon$, for ε small enough. We use a fixed point argument to show that this equation possesses a 2π-periodic solution for sufficiently small μ, which then gives the desired result.

By Duhamel's formula, the solution $v(\phi)$, for initial data $v(0) = v_0$, of the differential equation (3.17) satisfies the integral equation

$$v(\phi) = e^{-\frac{2a_r\mu}{\omega}\phi}v_0 + \int_0^\phi e^{-\frac{2a_r\mu}{\omega}(\phi-\theta)}\rho_1(v,\theta,\mu)d\theta.$$

The uniqueness of the solution of the initial value problem, and its differentiability with respect to the initial data v_0, allow us to conclude that, for $|v_0| < \varepsilon$, we have

$$v(\phi) = e^{-\frac{2a_r\mu}{\omega}\phi}v_0 + h(v_0,\phi,\mu), \quad h(v_0,\mu) = O(\mu v_0^2 + \mu^2)$$

for $\phi \in [0, 2\pi]$, where the function h is continuously differentiable. Now, if we can find a solution v_0 for the equation

$$v_0 = e^{-\frac{2a_r\mu}{\omega}2\pi}v_0 + h(v_0, 2\pi, \mu), \tag{3.18}$$

then the corresponding solution of the integral equation satisfies $v(2\pi) = v_0$, so that we have a *periodic orbit* of (3.12) in a small neighborhood of the circle $|A| = \mu^{1/2}\sqrt{-a_r/b_r}$. Indeed, observe that the *Poincaré map*

$$v_0 \mapsto e^{-\frac{2a_r\mu}{\omega}2\pi}v_0 + h(v_0, 2\pi, \mu),$$

is a contraction in a sufficiently small interval $[-\varepsilon, \varepsilon]$, because the derivative of the right hand side with respect to v_0 is

$$1 - \frac{2a_r\mu}{\omega}2\pi + O(\mu^2 + \mu\varepsilon) < 1$$

for ε and $\mu > 0$ small enough. Consequently, this mapping possesses a unique fixed point v_0 solution of (3.18) for sufficiently small $\mu > 0$.

This shows that the full equation (3.12) has a periodic orbit close to the circle of radius $|A| = r^*(\mu)$, and with period approximated by that of the solution (3.16) of the truncated equation. In addition, this proof allows us to conclude that this periodic orbit is attractive for $b_r < 0$. We summarize this result in the following *Hopf bifurcation theorem* (see also Figure 3.1).

Theorem 3.6 (Hopf bifurcation). *Assume that Hypotheses 3.1 and 3.5 hold. Then, for the differential equation (3.1) a supercritical (resp., subcritical) Hopf bifurcation occurs at $\mu = 0$ when $b_r < 0$ (resp., $b_r > 0$). More precisely, the following properties hold in a neighborhood of* 0 *in* $\mathbb{R}^2$ *for sufficiently small* μ*:*

(i) If $a_r b_r < 0$ (resp., $a_r b_r > 0$) the differential equation has precisely one equilibrium $u(\mu)$ for $\mu < 0$ (resp., for $\mu > 0$) with $u(0) = 0$. This equilibrium is stable when $b_r < 0$ and unstable when $b_r > 0$.

(ii) If $a_r b_r < 0$ (resp., $a_r b_r > 0$), the differential equation possesses for $\mu > 0$ (resp., for $\mu < 0$) an equilibrium $u(\mu)$ and a unique periodic orbit $u^*(\mu) = O(|\mu|^{1/2})$, which surrounds this equilibrium. The periodic orbit is stable when $b_r < 0$ and unstable when $b_r > 0$, whereas the equilibrium has opposite stability.

Remark 3.7. *The proof in dimension 2 is originally due to Andronov [1]. The n dimensional case is due to Hopf [24]. The present proof using normal form arguments is contained in Ruelle and Takens [62]. We also refer to Marsden and McCracken [51] and Vanderbauwhede [70].*

Remark 3.8 (Higher orders)**.** *In the above proof, we extensively use the assumption that the coefficient b_r is not zero. In the case when this coefficient is zero, one needs to consider the higher order terms, like the term of order $O(A|A|^4)$ in the expansion of the amplitude equation, and so on. If the problem is not completely degenerated, it is then possible to adapt the above proof without difficulty. Of course, this then gives other orders of magnitude for the bifurcating periodic solutions. We see in Chapter 4 of [21] that in the case of* reversible systems *all terms in Q_r in the radial equation disappear, leading to a degenerated situation.*

How to Compute the Hopf Bifurcation

We show now how to compute the important coefficients a and b in the normal form (3.12), (3.13), starting from the expansion of the vector field $\mathbf{F}$ in (3.1).

Consider the Taylor expansion of the vector field $\mathbf{F}$ in (3.1),

$$\mathbf{F}(u,\mu) = \sum_{1\le r+q\le k} \mu^q \mathbf{F}_{rq}(u^{(r)}) + o(|\mu| + ||u||)^k, \quad \mathbf{L} = \mathbf{F}_{10}, \tag{3.19}$$

where $\mathbf{F}_{rq}$ is the r-linear symmetric operator from $(\mathbb{R}^2)^r$ to $\mathbb{R}^2$,

$$\mathbf{F}_{rq} = \frac{1}{r!q!}\frac{\partial^q}{\partial\mu^q} D_u^r \mathbf{F}(0,0),$$

and $u^{(r)} := (u, u, ...u)$ for $u \in \mathbb{R}^2$. In particular, the map $u \mapsto \mathbf{F}_{rq}(u^{(r)})$ is homogeneous of degree r in the coordinates of u. Similarly, for $\mathbf{\Phi}_\mu$ in (3.6) we write

$$\mathbf{\Phi}_\mu(A,\overline{A}) = \sum_{1\le r+s+q\le p} \mathbf{\Phi}_{rsq} A^r \overline{A}^s \mu^q, \tag{3.20}$$

with

$$\mathbf{\Phi}_{100} = 0, \quad \mathbf{\Phi}_{010} = 0, \quad \mathbf{\Phi}_{rsq} = \overline{\mathbf{\Phi}}_{srq}.$$

Next, we substitute the change of variables (3.6) into (3.1) and obtain the identity

$$(\zeta + \partial_A \mathbf{\Phi}_\mu)\frac{dA}{dt} + (\overline{\zeta} + \partial_{\overline{A}} \mathbf{\Phi}_\mu)\frac{d\overline{A}}{dt} = \mathbf{F}(A\zeta + \overline{A\zeta} + \mathbf{\Phi}_\mu, \mu),$$

in which, according to the normal form (3.12)–(3.13), we have

$$\frac{dA}{dt} = i\omega A + a\mu A + bA|A|^2 + O(\mu^2|A| + |\mu||A|^3 + |A|^5). \tag{3.21}$$

Replacing $\mathbf{F}$ and $\mathbf{\Phi}_\mu$ by the expressions (3.19) and (3.20), we now identify the different powers of $(A, \overline{A}, \mu)$ in the identity above in order to determine the coefficients a, b, and $\mathbf{\Phi}_{rsq}$ from the known coefficients $\mathbf{F}_{rq}$.

First, at order $O(A)$ we recover the eigenvalue problem

$$i\omega\zeta = \mathbf{L}\zeta,$$

and then successively, respectively at orders $O(\mu)$, $O(\mu A)$, $O(A^2)$, $O(A\overline{A})$, $O(A^3)$, $O(A^2\overline{A})$, we find

$$\begin{aligned}
0 &= \mathbf{L}\mathbf{\Phi}_{001} + \mathbf{F}_{01} && (3.22)\\
a\zeta + (i\omega - \mathbf{L})\mathbf{\Phi}_{101} &= \mathbf{F}_{11}\zeta + 2\mathbf{F}_{20}(\zeta, \mathbf{\Phi}_{001}) && (3.23)\\
(2i\omega - \mathbf{L})\mathbf{\Phi}_{200} &= \mathbf{F}_{20}(\zeta, \zeta) && (3.24)\\
-\mathbf{L}\mathbf{\Phi}_{110} &= 2\mathbf{F}_{20}(\zeta, \overline{\zeta}) && (3.25)\\
(3i\omega - \mathbf{L})\mathbf{\Phi}_{300} &= 2\mathbf{F}_{20}(\zeta, \mathbf{\Phi}_{200}) + \mathbf{F}_{30}(\zeta, \zeta, \zeta) && (3.26)\\
b\zeta + (i\omega - \mathbf{L})\mathbf{\Phi}_{210} &= 2\mathbf{F}_{20}(\overline{\zeta}, \mathbf{\Phi}_{200}) + 2\mathbf{F}_{20}(\zeta, \mathbf{\Phi}_{110}) + 3\mathbf{F}_{30}(\zeta, \zeta, \overline{\zeta}). && (3.27)
\end{aligned}$$

All these equations are linear, and (3.22), (3.24), (3.25), (3.26) can be easily solved, because the operators $\mathbf{L}, (2i\omega - \mathbf{L}), (3i\omega - \mathbf{L})$ are invertible. This allows us to compute $\mathbf{\Phi}_{001}, \mathbf{\Phi}_{200}, \mathbf{\Phi}_{110}, \mathbf{\Phi}_{300}$, and the complex conjugates $\mathbf{\Phi}_{020}, \mathbf{\Phi}_{030}$. The equations (3.23) and (3.27) have the same structure, however, with the noninvertible matrix $(i\omega - \mathbf{L})$. The kernel of this matrix is one-dimensional, since $\pm i\omega$ are simple eigenvalues of $\mathbf{L}$, and one compatibility condition is needed in order to solve each of these equations. A convenient way of computing this compatibility condition is with the help of the eigenvector ζ^* of the adjoint operator satisfying

$$(-i\omega - \mathbf{L}^*)\zeta^* = 0, \quad \langle \zeta, \zeta^* \rangle = 1,$$

where $\langle \cdot, \cdot \rangle$ denotes the Hermitian scalar product in $\mathbb{C}^2$. (For $\zeta = (z_1, z_2) \in \mathbb{C}^2$ and $\zeta^* = (z_1^*, z_2^*) \in \mathbb{C}^2$, we take the Hermitian scalar product defined by

$$\langle \zeta, \zeta^* \rangle = z_1\overline{z_1^*} + z_2\overline{z_2^*}.)$$

Upon computing the Hermitian scalar product of these equations with ζ^* we find

$$a = \langle \mathbf{F}_{11}\zeta + 2\mathbf{F}_{20}(\zeta, \mathbf{\Phi}_{001}), \zeta^* \rangle, \tag{3.28}$$

and

$$b = \langle 2\mathbf{F}_{20}(\overline{\zeta}, \mathbf{\Phi}_{200}) + 2\mathbf{F}_{20}(\zeta, \mathbf{\Phi}_{110}) + 3\mathbf{F}_{30}(\zeta, \zeta, \overline{\zeta}), \zeta^* \rangle, \tag{3.29}$$

in which

$$\begin{aligned}
\mathbf{\Phi}_{001} &= -\mathbf{L}^{-1}\mathbf{F}_{01},\\
\mathbf{\Phi}_{200} &= (2i\omega - \mathbf{L})^{-1}\mathbf{F}_{20}(\zeta, \zeta),\\
\mathbf{\Phi}_{110} &= -2\mathbf{L}^{-1}\mathbf{F}_{20}(\zeta, \overline{\zeta}),
\end{aligned}$$

are obtained as explained above. We point out that $\mathbf{\Phi}_{001} = 0$ in the case when $u = 0$ is a solution for all μ, since then $\mathbf{F}(0, \mu) = 0$, so that $\mathbf{F}_{01} = 0$. In the same way, it is possible to derive formulas for higher order coefficients in (3.21), if needed.

3.2 Example: Homogeneous Brusselator

Consider the following system of ODEs:

$$\begin{aligned} \frac{du_1}{dt} &= -(\beta + 1)u_1 + u_1^2 u_2 + \alpha \\ \frac{du_2}{dt} &= \beta u_1 - u_1^2 u_2, \end{aligned} \tag{3.30}$$

in which $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^2$ and α, β are positive constants.

Remark 3.9. *This system, called the* homogeneous Brusselator *[60], arises in the modeling of an autocatalytic chemical reaction ruled by the following reaction mechanism:*

$$\begin{aligned} A &\xrightarrow{k_1} X \\ B + X &\xrightarrow{k_2} Y + D \\ 2X + Y &\xrightarrow{k_3} 3X \\ X &\xrightarrow{k_4} E. \end{aligned}$$

Here A, B, D, and E denote different chemical species, X and Y are intermediate products, and k_j represent the speeds of reactions. Denoting by X, Y, A, B the chemical concentrations of the corresponding species, assuming that the concentrations are homogeneous, and that the concentrations of components A and B are maintained constant, one finds that the evolution of X and Y is governed by the system of ODEs

$$\begin{aligned} \frac{dX}{dt} &= k_1 A - k_2 B X + k_3 X^2 Y - k_4 X \\ \frac{dY}{dt} &= k_2 B X - k_3 X^2 Y. \end{aligned}$$

Upon setting

$$u_1 = \sqrt{\frac{k_3}{k_4}} X, \quad u_2 = \sqrt{\frac{k_3}{k_4}} Y, \quad \alpha = \sqrt{\frac{k_3}{k_4}} \frac{k_1 A}{k_4}, \quad \beta = \frac{k_2 B}{k_4}, \quad \bar{t} = k_4 t,$$

this leads to the system (3.30), in which we have dropped the bar on t.

The system (3.30) possesses one equilibrium at $(u_1, u_2) = (\alpha, \beta/\alpha)$ for any positive constants α and β. The linearization at this equilibrium has the two eigenvalues

$$\lambda_{\pm} = \frac{1}{2}(\beta - 1 - \alpha^2) \pm \left(-\alpha^2 - \frac{1}{4}(\beta - 1 - \alpha^2)^2 \right)^{1/2}.$$

When $\beta < 1 + \alpha^2$, the equilibrium is stable, and it loses its stability at $\beta = 1 + \alpha^2$. At this point, the two eigenvalues are purely imaginary, $\lambda_{\pm} = \pm i\alpha$, and we are in the presence of a Hopf bifurcation.

Computation of the Hopf Bifurcation

In the system (3.30) we set

$$u_1 = \alpha + v_2, \quad u_2 = \frac{\beta}{\alpha} - (v_1 + v_2),$$

and

$$\omega = \alpha, \quad 2\mu = \beta - 1 - \alpha^2.$$

This leads to the system

$$\begin{aligned} \frac{dv_1}{dt} &= v_2 \\ \frac{dv_2}{dt} &= -\omega^2 v_1 + 2\mu v_2 - 2\omega v_1 v_2 + \frac{2\mu + 1 - \omega^2}{\omega} v_2^2 - (v_1 + v_2) v_2^2, \end{aligned} \tag{3.31}$$

in which ω is fixed, μ is a small bifurcation parameter, and $(0,0)$ is a solution for all values of ω and μ. The system (3.31) is of the form

$$\frac{dv}{dt} = \mathbf{L}v + \mathbf{R}(v, \mu), \tag{3.32}$$

where $v(t) = (v_1(t), v_2(t)) \in \mathbb{R}^2$ and

$$\mathbf{L} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad \mathbf{R}(v, \mu) = \mu \mathbf{R}_{11} v + \mathbf{R}_{20}(v, v) + \mu \mathbf{R}_{21}(v, v) + \mathbf{R}_{30}(v, v, v),$$

with

$$\mathbf{R}_{11} v = \begin{pmatrix} 0 \\ 2v_2 \end{pmatrix}, \quad \mathbf{R}_{21}(u, v) = \begin{pmatrix} 0 \\ \frac{2}{\omega} u_2 v_2 \end{pmatrix},$$

$$\mathbf{R}_{20}(u, v) = \begin{pmatrix} 0 \\ -\omega(u_1 v_2 + v_1 u_2) + \frac{1-\omega^2}{\omega} u_2 v_2 \end{pmatrix},$$

$$\mathbf{R}_{30}(u, v, w) = \begin{pmatrix} 0 \\ -\frac{1}{3}(u_1 v_2 w_2 + u_2 v_1 w_2 + u_2 v_2 w_1) - u_2 v_2 w_2 \end{pmatrix}.$$

Now, the linear operator $\mathbf{L}$ has the pair of simple purely imaginary eigenvalues $\pm i\omega$ with the associated eigenvectors

$$\zeta = \begin{pmatrix} 1 \\ i\omega \end{pmatrix}, \quad \overline{\zeta} = \begin{pmatrix} 1 \\ -i\omega \end{pmatrix}.$$

According to the results in the previous section the system (3.32) has the normal form (3.12). We are interested in computing the coefficients a and b in the expansion (3.13) of the polynomial Q. Of course we can use directly the formulas (3.28) and (3.29) for the coefficients a and b, but for the sake of clarity we prefer to go through the steps of the calculation, again.

Since we restrict ourselves to the terms of order 3 in the expansion of the normal form, it is enough to take $p = 3$ in the expansion (3.20). Then $\mathbf{\Phi}_\mu$ is a polynomial of degree 3,

$$\mathbf{\Phi}_\mu(A, \overline{A}) = \sum_{1 \le p+q+r \le 3} \mathbf{\Phi}_{pqr} A^p \overline{A}^q \mu^r, \quad \mathbf{\Phi}_{100} = \mathbf{\Phi}_{010} = 0,$$

such that the change of variables

$$v = A\zeta + \overline{A\zeta} + \mathbf{\Phi}_\mu(A, \overline{A}) \tag{3.33}$$

transforms (3.31) into the normal form

$$\frac{dA}{dt} = i\omega A + a\mu A + bA|A|^2 + O(|A|(|\mu|^2 + |\mu||A|^2 + |A|^3)). \tag{3.34}$$

By arguing as explained in the previous section, i.e., substituting (3.33) in (3.32), then replacing dA/dt from (3.34), and finally identifying the different powers of $(A, \overline{A}, \mu)$, we find the system (3.22)–(3.27) with $\mathbf{F}_{ij} = \mathbf{R}_{ij}$. Since $\mathbf{R}_{01} = 0$, we have $\mathbf{\Phi}_{001} = 0$, and the identity (3.23) becomes

$$a\zeta + (i\omega - \mathbf{L})\mathbf{\Phi}_{101} = \mathbf{R}_{11}\zeta = \begin{pmatrix} 0 \\ 2i\omega \end{pmatrix}.$$

The coefficient a is now found from the solvability condition for this equation, obtained by taking the Hermitian scalar product with the vector ζ^* in the kernel of the adjoint operator satisfying

$$(-i\omega - \mathbf{L}^*)\zeta^* = 0, \quad \langle \zeta, \zeta^* \rangle = 1.$$

A direct calculation shows that

$$\mathbf{L}^* = \begin{pmatrix} 0 & -\omega^2 \\ 1 & 0 \end{pmatrix}, \quad \zeta^* = \frac{1}{2i\omega} \begin{pmatrix} i\omega \\ -1 \end{pmatrix},$$

and then

$$a = \langle \mathbf{R}_{11}\zeta, \zeta^* \rangle = 1.$$

Remark 3.10. *Since $\mathbf{R}(0, \mu) = 0$, it is not difficult to check in this case that the eigenvalues of the 2×2-matrix*

$$\mathbf{L} + \mu\mathbf{R}_{11} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 2\mu \end{pmatrix},$$

obtained by linearizing the system (3.32) at $U = 0$, are the same as the eigenvalues of the 2×2-matrix obtained by linearizing the normal form equation (3.34), together with the complex conjugated equation, at $(A, \overline{A}) = (0, 0)$. We can use this property to compute the coefficient a in a different way. Indeed, this latter matrix is of the form

$$\begin{pmatrix} i\omega + a\mu & 0 \\ 0 & -i\omega + \overline{a}\mu \end{pmatrix} + O(\mu^2),$$

and since the eigenvalues of $\mathbf{L} + \mu\mathbf{R}_{11}$ are

$$\lambda_\pm = \mu \pm i\sqrt{\omega^2 - \mu^2} = \pm i\omega + \mu \mp \frac{i\mu^2}{2\omega} + O(\mu^4),$$

we can conclude that $a = 1$.

Next, in order to compute the coefficient b we use the equations (3.24), (3.25), and (3.27), i.e.,

$$\begin{aligned}(2i\omega - \mathbf{L})\mathbf{\Phi}_{200} &= \mathbf{R}_{20}(\zeta,\zeta),\\ -\mathbf{L}\mathbf{\Phi}_{110} &= 2\mathbf{R}_{20}(\zeta,\overline{\zeta}),\\ b\zeta + (i\omega - \mathbf{L})\mathbf{\Phi}_{210} &= 2\mathbf{R}_{20}(\overline{\zeta},\mathbf{\Phi}_{200}) + 2\mathbf{R}_{20}(\zeta,\mathbf{\Phi}_{110}) + 3\mathbf{R}_{30}(\zeta,\zeta,\overline{\zeta}).\end{aligned}$$

Solving the first two equations we find

$$\mathbf{\Phi}_{200} = \begin{pmatrix} \frac{1-\omega^2}{3\omega} + \frac{2i}{3} \\ -\frac{4\omega}{3} + \frac{2i}{3}(1-\omega^2) \end{pmatrix}, \quad \mathbf{\Phi}_{110} = \begin{pmatrix} \frac{2(1-\omega^2)}{\omega} \\ 0 \end{pmatrix},$$

and then

$$2\mathbf{R}_{20}(\overline{\zeta},\mathbf{\Phi}_{200}) = \begin{pmatrix} 0 \\ \frac{4}{3}(1-\omega^2+\omega^4) + 2i\omega(1-\omega^2) \end{pmatrix},$$

$$2\mathbf{R}_{20}(\zeta,\mathbf{\Phi}_{110}) = \begin{pmatrix} 0 \\ -4i\omega(1-\omega^2) \end{pmatrix}, \quad 3\mathbf{R}_{30}(\zeta,\zeta,\overline{\zeta}) = \begin{pmatrix} 0 \\ -\omega^2 - 3i\omega^3 \end{pmatrix}.$$

Finally, we compute b from the solvability condition for the third equation,

$$b = \langle 2\mathbf{R}_{20}(\overline{\zeta},\mathbf{\Phi}_{200}) + 2\mathbf{R}_{20}(\zeta,\mathbf{\Phi}_{110}) + 3\mathbf{R}_{30}(\zeta,\zeta,\overline{\zeta}),\zeta^*\rangle.$$

We find

$$\begin{aligned}&2\mathbf{R}_{20}(\overline{\zeta},\mathbf{\Phi}_{200}) + 2\mathbf{R}_{20}(\zeta,\mathbf{\Phi}_{110}) + 3\mathbf{R}_{30}(\zeta,\zeta,\overline{\zeta})\\ &\qquad = \begin{pmatrix} 0 \\ \frac{1}{3}(4-7\omega^2+4\omega^4) - i\omega(2+\omega^2) \end{pmatrix},\end{aligned}$$

which gives

$$b = -\frac{1}{2}(2+\omega^2) - \frac{i}{6\omega}(4-7\omega^2+4\omega^4). \tag{3.35}$$

In particular, this shows that the real part b_r of b is negative, so that we have here a *supercritical Hopf bifurcation*.

3.3 Hopf Bifurcation with $SO(2)$ Symmetry

We discuss in this section a particular case of a Hopf bifurcation, where the vector field possesses a continuous symmetry. As before, we assume that the vector field $\mathbf{F}$ in (3.1) satisfies Hypotheses 3.1 and 3.5, and now further assume that the following holds.

Hypothesis 3.11. *We assume that the vector field* $\mathbf{F}$ *is* $SO(2)$*-equivariant, that is, there exists a one-parameter continuous family of linear maps* $\mathbf{R}_\varphi$ *on* $\mathbb{R}^2$*, for* $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$*, with the following properties:*

(i) $\mathbf{R}_\varphi \circ \mathbf{R}_\psi = \mathbf{R}_{\varphi+\psi}$ *for all* $\varphi,\ \psi \in \mathbb{R}/2\pi\mathbb{Z}$*;*

(ii) $\mathbf{R}_0 = \mathbb{I}$*;*

(iii) $\mathbf{F}(\mathbf{R}_\varphi u, \mu) = \mathbf{R}_\varphi \mathbf{F}(u, \mu)$ *for all* $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$.

An immediate consequence of the third property in this hypothesis is that if $u(\mu)$ is a steady solution of (3.1), then $\mathbf{R}_\varphi u(\mu)$ is also a steady solution of (3.1). On the other hand, as already noticed in the Remark 3.2, the system (3.1) has a unique steady solution in a neighborhood of the origin for all sufficiently small μ. Then we necessarily have $\mathbf{R}_\varphi u(\mu) = u(\mu)$, that is, *the steady solution* $u(\mu)$ *is invariant under the action of* $\mathbf{R}_\varphi$. In addition, notice that

$$\mathbf{L}(\mathbf{R}_\varphi \zeta) = \mathbf{R}_\varphi(\mathbf{L}\zeta) = i\omega(\mathbf{R}_\varphi \zeta),$$

and since the eigenvalue $i\omega$ is simple we have

$$\mathbf{R}_\varphi \zeta = k(\varphi)\zeta \text{ for some } k(\varphi) \in \mathbb{C}.$$

Using the group properties of $\mathbf{R}_\varphi$, Hypothesis 3.11(i)–(ii), we obtain that $k(\varphi + \psi) = k(\varphi)k(\psi)$ for all φ, ψ, and that $k(0) = 1$. The fact that k is a continuous function of $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, now implies that

$$k(\varphi) = e^{im\varphi}, \quad m \in \mathbb{Z}. \tag{3.36}$$

We now distinguish two cases depending upon the value of m in (3.36).

First, assume that $m = 0$, which means that the action of the group $\mathbf{R}_\varphi$ on the eigenvector ζ is trivial, $\mathbf{R}_\varphi \zeta = \zeta$. Then the same also holds for the complex conjugated eigenvector $\overline{\zeta}$, and since $\{\zeta, \overline{\zeta}\}$ forms a basis of $\mathbb{R}^2$, we have in this case $\mathbf{R}_\varphi = \mathbb{I}$ for all φ. Consequently, the action of the continuous group $\mathbf{R}_\varphi$ is trivial, so that there is no new fact with respect to Theorem 3.6 in this case, except that all points of the periodic bifurcating orbit are invariant under $\mathbf{R}_\varphi$.

Next, assume that $m \neq 0$. Then in the basis $\{\zeta, \overline{\zeta}\}$ of $\mathbb{R}^2$, the action of $\mathbf{R}_\varphi$ on the coordinates $(z, \overline{z})$, $z \in \mathbb{C}$ is given by

$$\mathbf{R}_\varphi = \begin{pmatrix} e^{im\varphi} & 0 \\ 0 & e^{-im\varphi} \end{pmatrix}.$$

This matrix commutes now with the vector field in equation (3.5), so that we have

$$f(e^{im\varphi}z, e^{-im\varphi}\overline{z}, \mu) = e^{im\varphi} f(z, \overline{z}, \mu)$$

for all $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ and all z in a neighborhood of 0. Then, by Lemma 3.4, it follows that the differential equation (3.5) is of the form

$$\frac{dz}{dt} = i\omega z + zg(|z|, \mu), \tag{3.37}$$

with g of class $\mathcal{C}^{k-1}$ and even in $|z|$. This means that in this case the *equation is already in the normal form* (3.12), with polynomial Q given by the regular part in the Taylor expansion of g, and the rest, ρ, being of the form z times a function depending only upon $|z|$. The particular form of this part allows to use the same arguments as for the truncated normal form and to show that in this case for the bifurcating periodic solutions $u^*(\cdot; \mu)$ the coordinate $z^*(\cdot,\mu)$ is of the form (3.16). In particular, they describe a "circle" in the plane $\mathbb{C}$. Furthermore, from (3.16) we obtain

$$\mathbf{R}_\varphi u(t;\mu) = u(t + \frac{m\varphi}{\omega^*(\mu)};\mu).$$

Choosing $\varphi = -\omega^*(\mu)t/m$, we obtain

$$\mathbf{R}_{-\omega^*(\mu)t/m} u(t;\mu) = u(0;\mu),$$

and this gives a new formula for the periodic solutions,

$$u(t;\mu) = \mathbf{R}_{\frac{\omega^*(\mu)t}{m}} u(0;\mu). \tag{3.38}$$

These periodic solutions are *rotating waves*, with wavenumber m thanks to the property

$$u(t;\mu) = \mathbf{R}_{\frac{2\pi}{m}} u(t;\mu).$$

This proves the following result:

Corollary 3.12 (Hopf bifurcation with $SO(2)$ symmetry)**.** *Assume that Hypotheses 3.1, 3.5, and 3.11 hold. Further assume that the action of the group $\mathbf{R}_\varphi$ is not trivial. Then the family of periodic solutions bifurcating in the Hopf bifurcation at $\mu = 0$ are the* rotating waves *(3.38), with wavenumber m given by the action of the group on the eigenvector ζ of $\mathbf{L}$ associated with the purely imaginary eigenvalue $i\omega$.*

3.4 Steady Bifurcation with $O(2)$ Symmetry

We end this section with a case where the differential equation (3.1) possesses a one-parameter group of symmetries together with one discrete symmetry. More precisely, we make the following assumption.

Hypothesis 3.13. *Assume that the vector field $\mathbf{F}$ in (3.1) is of class $\mathcal{C}^k$, $k \geq 3$, that it satisfies (3.2), and that 0 is an eigenvalue of $\mathbf{L}$. Further assume that $\mathbf{F}$ is $O(2)$-equivariant, that is, there exists a one-parameter continuous family of linear maps $\mathbf{R}_\varphi$ on $\mathbb{R}^2$, for $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, and a symmetry $\mathbf{S}$ on $\mathbb{R}^2$ with the following properties:*

- *(i)* $\mathbf{R}_\varphi \circ \mathbf{R}_\psi = \mathbf{R}_{\varphi+\psi}$ *and* $\mathbf{S}\mathbf{R}_\varphi = \mathbf{R}_{-\varphi}\mathbf{S}$ *for all* $\varphi,\ \psi \in \mathbb{R}/2\pi\mathbb{Z}$*;*
- *(ii)* $\mathbf{R}_0 = \mathbb{I}$ *and* $\mathbf{S}^2 = \mathbb{I}$*;*
- *(iii)* $\mathbf{F}(\mathbf{R}_\varphi u,\mu) = \mathbf{R}_\varphi \mathbf{F}(u,\mu)$ *and* $\mathbf{F}(\mathbf{S}u,\mu) = \mathbf{S}\mathbf{F}(u,\mu)$ *for all* $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$*.*

Remark 3.14. *This type of symmetry is very frequent in physical examples, particularly in systems of PDEs (infinite-dimensional case) when the system is invariant under translations in one unbounded spatial direction and possesses a reflection symmetry in this direction. When looking for solutions that are periodic in this unbounded spatial direction, the invariance under spatial translations provides the one-parameter group of symmetries, whereas the reflection is the discrete symmetry. We present an example of such a PDE in Section 4.4.2.*

An important consequence of the $O(2)$-equivariance in this hypothesis is that any eigenvalue of the linear map $\mathbf{L}$ is double, provided the action of the group $\mathbf{R}_\varphi$ is not trivial. Indeed, any eigenvalue of $\mathbf{L}$ is either simple or double. Assume that $\lambda \in \mathbb{C}$ is a simple eigenvalue of $\mathbf{L}$, with associated eigenvector ζ. Then we have

$$\mathbf{L}(\mathbf{R}_\varphi \zeta) = \mathbf{R}_\varphi(\mathbf{L}\zeta) = \lambda(\mathbf{R}_\varphi \zeta),$$

so that $\mathbf{R}_\varphi \zeta = r(\varphi)\zeta$ for some $r(\varphi) \in \mathbb{C}$, and similarly $\mathbf{S}\zeta = s\zeta$ for some $s \in \mathbb{C}$. As for $k(\varphi)$ given by (3.36), in the case of the Hopf bifurcation with $SO(2)$ symmetry discussed in the previous section, we conclude that $r(\varphi) = e^{im\varphi}$, with $m \in \mathbb{Z}$. Moreover, since $\mathbf{S}^2 = \mathbb{I}$, we have that $s = \pm 1$, and from the equality $\mathbf{R}_\varphi \mathbf{S}\zeta = \mathbf{S}\mathbf{R}_{-\varphi}\zeta$, we obtain that $se^{im\varphi}\zeta = se^{-im\varphi}\zeta$ for all φ. Thus $m = 0$, so that $\mathbf{R}_\varphi = \mathbb{I}$ for all φ, which means that the group represented by $\mathbf{R}_\varphi$ reduces to the identity. Consequently, if the action of the group $\mathbf{R}_\varphi$ is not trivial, then λ is a double eigenvalue of $\mathbf{L}$. We shall therefore make the following hypothesis.

Hypothesis 3.15. *Assume that zero is a double eigenvalue of* $\mathbf{L}$ *and that the action of* $\mathbf{R}_\varphi$ *on* $\mathbb{R}^2$ *is not trivial.*

Now we construct a suitable basis for $\mathbb{R}^2$ in which the action of $\mathbf{R}_\varphi$ and $\mathbf{S}$ is given by the 2×2-matrices

$$\mathbf{R}_\varphi = \begin{pmatrix} e^{im\varphi} & 0 \\ 0 & e^{-im\varphi} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.39}$$

First, we claim that the eigenvectors of $\mathbf{R}_\varphi$ are independent of φ, and more precisely, that an eigenvector ζ_0 of $\mathbf{R}_{\varphi_0}$ for some φ_0 is also an eigenvector of $\mathbf{R}_\varphi$ for any φ, namely,

$$\mathbf{R}_\varphi \zeta_0 = r(\varphi)\zeta_0, \tag{3.40}$$

with corresponding eigenvalue $r(\varphi)$ depending continuously upon φ such that

$$r(0) = 1, \quad r(\varphi + \psi) = r(\varphi)r(\psi) \text{ for all } \varphi, \psi \in \mathbb{R}/2\pi\mathbb{Z}. \tag{3.41}$$

This result is proved in [21].

An immediate consequence of the continuity of $r(\varphi)$ in φ and of the equalities (3.41) is that $r(\varphi) = e^{im\varphi}$ for some $m \in \mathbb{R}$ for all φ. Here $m \neq 0$, because $\mathbf{R}_\varphi$ acts nontrivially on $\mathbb{R}^2$. Next, $\mathbf{R}_\varphi \mathbf{S}\zeta_0 = e^{-im\varphi}\mathbf{S}\zeta_0$, so that $\mathbf{S}\zeta_0$ is an eigenvector of $\mathbf{R}_\varphi$ for the eigenvalue $e^{-im\varphi}$. Together with ζ_0, which is an eigenvector of the same operator $\mathbf{R}_\varphi$ for the eigenvalue $e^{im\varphi}$, this provides us with a basis for $\mathbb{R}^2$. In particular, there exists $k \in \mathbb{C}$ such that

$$\mathbf{S}\zeta_0 = k\overline{\zeta_0},$$

and the property $\mathbf{S}^2 = \mathbb{I}$ leads to $|k| = 1$, i.e.,

$$k = e^{i\beta}, \quad \beta \in \mathbb{R}.$$

We set

$$\zeta = e^{-i\beta/2}\zeta_0,$$

for which we find

$$\mathbf{S}\zeta = e^{-i\beta/2}\mathbf{S}\zeta_0 = e^{i\beta/2}\overline{\zeta_0} = \overline{\zeta}.$$

It is then straightforward to conclude that the action of the operators $\mathbf{R}_\varphi$ and $\mathbf{S}$ in the basis $\{\zeta, \overline{\zeta}\}$ is given by (3.39).

We now proceed as for the Hopf bifurcation, and represent any $u \in \mathbb{R}^2$ by a complex coordinate $z \in \mathbb{C}$ through

$$u = z\zeta + \overline{z}\overline{\zeta}.$$

Similarly, for the vector field $\mathbf{F}$ we write

$$\mathbf{F}(u, \mu) = f(z, \overline{z}, \mu)\zeta + \overline{f}(z, \overline{z}, \mu)\overline{\zeta},$$

and then obtain two complex differential equations

$$\frac{dz}{dt} = f(z, \overline{z}, \mu)$$

and its complex conjugate. The equivariance properties in Hypothesis 3.13(iii) and the equalities in (3.39) imply that f satisfies the relations

$$f(e^{im\varphi}z, e^{-im\varphi}\overline{z}, \mu) = e^{im\varphi} f(z, \overline{z}, \mu),$$

and

$$f(\overline{z}, z, \mu) = \overline{f(z, \overline{z}, \mu)}$$

for all z and μ. Using Lemma 3.4, again, the first relation implies that

$$f(z, \overline{z}, \mu) = zg(|z|, \mu),$$

where g is a complex function of class $\mathcal{C}^{k-1}$ in a neighborhood of 0, and even in $|z|$. The second relation implies that, in addition, g is real-valued.

We introduce polar coordinates $A = re^{i\phi}$ and obtain the system

$$\frac{dr}{dt} = rg(r, \mu) = a\mu r + br^3 + o(r|\mu| + r^3) \tag{3.42}$$

$$\frac{d\phi}{dt} = 0, \tag{3.43}$$

in which the coefficients a and b are found from the Taylor expansion of g. Since the function g is even in r, the scalar vector field in (3.42) satisfies Hypothesis 2.6 from the case of a pitchfork bifurcation, provided the coefficients a and b are nonzero. We therefore assume now:

Hypothesis 3.16. *Assume that the coefficients a and b in (3.42) are nonzero,*

$$\frac{\partial g}{\partial \mu}(0, 0) =: a \neq 0, \quad \frac{\partial^2 g}{\partial r^2}(0, 0) =: 2b \neq 0.$$

Applying the result in Theorem 2.9, we conclude that for the equation (3.42) a *pitchfork bifurcation* occurs at $\mu = 0$, which is supercritical when $b < 0$ and subcritical when $b > 0$. The bifurcation diagrams for this equation are the same as those in Figure 2.3. Since for the radial equation we are restricted to positive solutions, this shows that for $ab < 0$ (resp., $ab > 0$), the radial equation possesses the positive steady solution

$$r^*(\mu) = \sqrt{-\frac{a\mu}{b}} + o(|\mu|^{3/2})$$

for $\mu > 0$ (resp., $\mu < 0$). The dynamics of the second equation (3.43) is trivial, showing that the phase ϕ of the solutions stays constant in time t.

Going back to the two-dimensional equation (3.1), this shows that at the bifurcation point $\mu = 0$, a "circle" of equilibria, parameterized by the phase ϕ,

$$u^*(\mu, \phi) = r^*(\mu)e^{i\phi}\zeta + r^*(\mu)e^{-i\phi}\overline{\zeta},$$

bifurcates for $\mu > 0$ (resp., $\mu < 0$) when $ab < 0$ (resp., $ab > 0$). We have here a steady bifurcation. The stability of the bifurcating equilibria is given by that of $r^*(\mu)$, so that they are stable when $b < 0$ and unstable when $b > 0$. Figure 3.2 illustrates the phase portraits for $\mu < 0$ and $\mu > 0$ in the case $a > 0$, $b < 0$. Similar phase portraits can be obtained in the other cases.

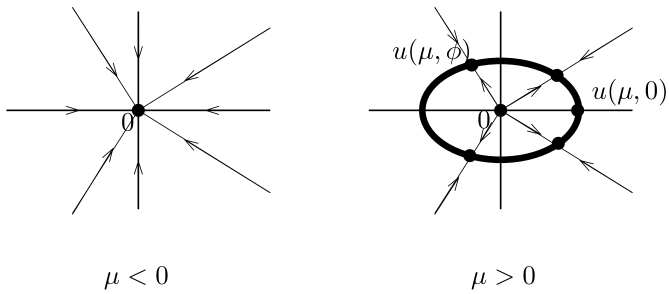


Figure 3.2: Steady bifurcation with $O(2)$ symmetry: phase portraits in the u-plane for equation (3.1) in the case $a > 0$ and $b < 0$.

In addition, we have that the bifurcating equilibria are invariant under the rotation $\mathbf{R}_{\frac{2\pi}{m}}$, since

$$\mathbf{R}_{\frac{2\pi}{m}} u^*(\mu, \phi) = u^*(\mu, \phi),$$

and there are two equilibria that are symmetric, i.e., invariant under the symmetry $\mathbf{S}$,

$$\mathbf{S}u^*(\mu, 0) = u^*(\mu, 0), \quad \mathbf{S}u^*(\mu, \pi) = u^*(\mu, \pi).$$

Moreover, $u^*(\mu, \phi)$ may be obtained from $u^*(\mu, 0)$ through

$$u^*(\mu, \phi) = R_{\frac{\phi}{m}} u^*(\mu, 0).$$

This shows that we have a *group orbit* of equilibria. We summarize these results in the following theorem.

Theorem 3.17 (Steady bifurcation with $O(2)$ symmetry)**.** *Assume that Hypotheses 3.13, 3.15, and 3.16 hold. Then, for the differential equation (3.1) a steady bifurcation occurs at $\mu = 0$. More precisely, the following properties hold in a neighborhood of* 0 *in* $\mathbb{R}^2$ *for sufficiently small μ:*

(i) If $ab < 0$ (resp., $ab > 0$) the differential equation has precisely one trivial equilibrium $u = 0$ for $\mu < 0$ (resp., for $\mu > 0$). This equilibrium is stable when $b < 0$ and unstable when $b > 0$.

(ii) If $ab < 0$ (resp., $ab > 0$), the differential equation possesses for $\mu > 0$ (resp., for $\mu < 0$), the equilibrium $u = 0$ and a unique closed orbit of equilibria $u^*(\mu, \phi) = O(|\mu|^{1/2})$ for $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, which surrounds this equilibrium. These equilibria are stable when $b < 0$ and unstable when $b > 0$, whereas the equilibrium $u = 0$ has opposite stability.

(iii) The equilibria $u^*(\mu, \phi)$ satisfy

$$u^*(\mu, \phi) = R_{\frac{\phi}{m}} u^*(\mu, 0),$$

they are all invariant under the action of $\mathbf{R}_{\frac{2\pi}{m}}$, and there are two equilibria, $u^*(\mu, 0)$ and $u^*(\mu, \pi)$, invariant under the symmetry $\mathbf{S}$.

Remark 3.18 (Higher orders)**.** *In the case where the coefficients a or (and) b in Hypothesis 3.16 vanish, one has to consider the next nonzero higher order terms in the expansion of g, just as in the case of the pitchfork bifurcation.*

4 Center manifolds

This section is devoted to center manifold theory. We present a general result on the existence of local center manifolds for infinite-dimensional systems in Section 4.2 and then discuss several particular cases and extensions, as, for instance, to parameter-dependent systems and systems possessing different symmetries in Section 4.3. We give a series of examples showing how these results apply to various situations in Section 4.4.

4.1 Notations

Consider two (complex or real) Hilbert spaces $\mathcal{X}$ and $\mathcal{Z}$. Throughout this chapter we shall use the following notations:

- $B_\varepsilon(\mathcal{X})$ is the closed ball $\{u \in \mathcal{X}; \|u\|_\mathcal{X} \leq \varepsilon\}$.
- $\mathcal{C}^k(\mathcal{Z}, \mathcal{X})$ is the Banach space of k-times continuously differentiable functions $F : \mathcal{Z} \to \mathcal{X}$ equipped with the sup norm on all derivatives up to order k,

$$\|F\|_{\mathcal{C}^k} = \max_{j=0,\dots,k} \left(\sup_{y \in \mathcal{Z}} \left(\|D^j F(y)\|_{\mathcal{L}(\mathcal{Z}^j, \mathcal{X})} \right) \right);$$

  here, and in the following, D denotes the differentiation operator.
- $\mathcal{L}(\mathcal{Z}, \mathcal{X})$ is the Banach space of linear bounded operators $\mathbf{L} : \mathcal{Z} \to \mathcal{X}$, equipped with the operator norm

$$\|\mathbf{L}\|_{\mathcal{L}(\mathcal{Z},\mathcal{X})} = \sup_{\|u\|_\mathcal{Z}=1} \left(\|\mathbf{L}u\|_\mathcal{X} \right).$$

  If $\mathcal{Z} = \mathcal{X}$, we write $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X})$.

- For a linear operator $\mathbf{L} : \mathcal{Z} \to \mathcal{X}$, we denote by $\operatorname{im} \mathbf{L}$ its *range*,

$$\operatorname{im} \mathbf{L} = \{\mathbf{L}u \in \mathcal{X} \; ; \; u \in \mathcal{Z}\} \subset \mathcal{X},$$

  and by $\ker \mathbf{L}$ its *kernel*,

$$\ker \mathbf{L} = \{u \in \mathcal{Z} \; ; \; \mathbf{L}u = 0\} \subset \mathcal{Z}.$$

- Assume that $\mathcal{Z} \hookrightarrow \mathcal{X}$ with continuous embedding. For a linear operator $\mathbf{L} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$ we denote by $\rho(\mathbf{L})$, or simply ρ, if there is no risk of confusion, the *resolvent set* of $\mathbf{L}$,

$$\rho = \{\lambda \in \mathbb{C} \; ; \; \lambda \mathbb{I} - \mathbf{L} : \mathcal{Z} \to \mathcal{X} \text{ is bijective}\}.$$

  The complement of the resolvent set is the *spectrum* $\sigma(\mathbf{L})$, or simply σ,

$$\sigma = \mathbb{C} \setminus \{\rho\}.$$

  Notice that when the operator $\mathbf{L}$ is real, the resolvent set and the spectrum of $\mathbf{L}$ are both symmetric with respect to the real axis in the complex plane.

4.2 Local Center Manifolds

In this section we present the main result on the existence of local center manifolds. We discuss the hypotheses in Section 4.2.1, and state the main theorem in Section 4.2.2. The proof of the theorem is given in [21].

4.2.1 Hypotheses

Let $\mathcal{X}$, $\mathcal{Z}$ be (real or complex) Hilbert spaces such that

$$\mathcal{Z} \hookrightarrow \mathcal{X},$$

with continuous embeddings. We consider a differential equation in $\mathcal{X}$ of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u), \tag{4.1}$$

in which we assume that the linear part $\mathbf{L}$ and the nonlinear part $\mathbf{R}$ are such that the following holds.

Hypothesis 4.1. *We assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in (4.1) have the following properties:*

(i) $\mathbf{L} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$*;*

(ii) for some $k \geq 2$*, there exists a neighborhood* $\mathcal{V} \subset \mathcal{Z}$ *of* 0 *such that* $\mathbf{R} \in \mathcal{C}^k(\mathcal{V}, \mathcal{X})$ *and*

$$\mathbf{R}(0) = 0, \quad D\mathbf{R}(0) = 0.$$

Remark 4.2. *The condition* $\mathbf{R}(0) = 0$ *means that* 0 *is an equilibrium of the differential equation (4.1), and the condition* $D\mathbf{R}(0) = 0$ *then shows that* $\mathbf{L}$ *is the linearization of the vector field about* 0*, so that* $\mathbf{R}$ *represents the nonlinear terms which are* $O(\|u\|_{\mathcal{Z}}^2)$*. More generally, for an equation which has a nonzero equilibrium,* u_**, say, we recover these conditions after replacing* u *by* $u - u_*$ *and then taking for* $\mathbf{L}$ *the differential of the resulting vector field at* 0.

Definition 4.3. *A* solution *of the differential equation (4.1) is a function* $u : \mathcal{I} \to \mathcal{Z} \hookrightarrow X$ *defined on an interval* $\mathcal{I} \subset \mathbb{R}$*, with the following properties:*

(i) the map $u : \mathcal{I} \to \mathcal{Z}$ *is continuous;*

(ii) the map $u : \mathcal{I} \to \mathcal{X}$ *is continuously differentiable;*

(iii) the equality (4.1) holds in $\mathcal{X}$ *for all* $t \in \mathcal{I}$.

Besides Hypothesis 4.1, we make two further assumptions on the linear operator $\mathbf{L}$, which are essential for the center manifold theorem.

Hypothesis 4.4 (Spectral decomposition)**.** *Consider the spectrum* σ *of the linear operator* $\mathbf{L}$*, and write*

$$\sigma = \sigma_+ \cup \sigma_0 \cup \sigma_-,$$

in which

$$\sigma_+ = \{\lambda \in \sigma \; ; \; Re\lambda > 0\}, \quad \sigma_0 = \{\lambda \in \sigma \; ; \; Re\lambda = 0\}, \quad \sigma_- = \{\lambda \in \sigma \; ; \; Re\lambda < 0\}.$$

We assume that

(i) σ_+ *is empty, and there exists a positive constant* $\gamma > 0$ such that $\sup_{\lambda \in \sigma_-} (Re\lambda) < -\gamma$ *;*

(ii) the set σ_0 *consists of a finite number of eigenvalues with finite algebraic multiplicities.*

Remark 4.5. *The hypothesis above implies that the resolvent set* ρ *of* $\mathbf{L}$ *is not empty. This further implies that* $\mathbf{L}$ *is a closed operator in* $\mathcal{X}$*. Indeed, for some* $\lambda \in \rho$*, the operator* $\lambda\mathbb{I} - \mathbf{L}$ *is bijective, and since* $\mathbb{I}$ *and* $\mathbf{L}$ *belong to* $\mathcal{L}(\mathcal{Z}, \mathcal{X})$*, by the closed graph theorem the resolvent* $(\lambda\mathbb{I} - \mathbf{L})^{-1}$ *belongs to* $\mathcal{L}(\mathcal{X}, \mathcal{Z})$*. Now* $\mathcal{L}(\mathcal{X}, \mathcal{Z}) \subset \mathcal{L}(\mathcal{X})$*, so that* $(\lambda\mathbb{I} - \mathbf{L})^{-1} \in \mathcal{L}(\mathcal{X})$ *and then by the closed graph theorem* $\lambda\mathbb{I} - \mathbf{L}$ *is closed in* $\mathcal{X}$*. Consequently,* $\mathbf{L}$ *is closed in* $\mathcal{X}$.

As a consequence of Hypothesis 4.4 (ii), we can define the (spectral) projection $\mathbf{P}_0 \in \mathcal{L}(\mathcal{X})$, corresponding to σ_0, by the Dunford integral formula

$$\mathbf{P}_0 = \frac{1}{2\pi i} \int_\Gamma (\lambda\mathbb{I} - \mathbf{L})^{-1} \mathrm{d}\lambda, \tag{4.2}$$

where Γ is a simple, oriented counterclockwise, Jordan curve surrounding σ_0 and lying entirely in $\{\lambda \in \mathbb{C} \; ; \; |Re\lambda| < \gamma\}$. Then

$$\mathbf{P}_0^2 = \mathbf{P}_0, \quad \mathbf{P}_0\mathbf{L}u = \mathbf{L}\mathbf{P}_0 u \text{ for all } u \in \mathcal{Z},$$

and the range $\operatorname{im} \mathbf{P}_0$ is finite-dimensional, since σ_0 consists of a finite number of eigenvalues with finite algebraic multiplicities. In particular, it satisfies $\operatorname{im} \mathbf{P}_0 \subset \mathcal{Z}$, and

$$\mathbf{P}_0 \in \mathcal{L}(\mathcal{X}, \mathcal{Z}),$$

since the map $\lambda \mapsto (\lambda \mathbb{I} - \mathbf{L})^{-1} \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ is analytic in a neighborhood of Γ.

We define a second projection $\mathbf{P}_h : \mathcal{X} \to \mathcal{X}$ by

$$\mathbf{P}_h = \mathbb{I} - \mathbf{P}_0,$$

which then also satisfies

$$\mathbf{P}_h^2 = \mathbf{P}_h, \quad \mathbf{P}_h \mathbf{L} u = \mathbf{L} \mathbf{P}_h u \text{ for all } u \in \mathcal{Z},$$

and

$$\mathbf{P}_h \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Z}),$$

since $\mathbf{P}_0 \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ and the embedding $\mathcal{Z} \hookrightarrow \mathcal{X}$ is continuous [1].

Next, we consider the spectral subspaces associated with these two projections,

$$\mathcal{E}_0 = \operatorname{im} \mathbf{P}_0 = \ker \mathbf{P}_h \subset \mathcal{Z}, \quad \mathcal{X}_h = \operatorname{im} \mathbf{P}_h = \ker \mathbf{P}_0 \subset \mathcal{X},$$

which provide a decomposition of $\mathcal{X}$ into invariant subspaces,

$$\mathcal{X} = \mathcal{E}_0 \oplus \mathcal{X}_h.$$

We also set

$$\mathcal{Z}_h = \mathbf{P}_h \mathcal{Z} \subset \mathcal{Z},$$

and denote by $\mathbf{L}_0$ and $\mathbf{L}_h$ the restrictions of $\mathbf{L}$ to $\mathcal{E}_0$ and $\mathcal{Z}_h$, respectively,

$$\mathbf{L}_0 \in \mathcal{L}(\mathcal{E}_0), \quad \mathbf{L}_h \in \mathcal{L}(\mathcal{Z}_h, \mathcal{X}_h).$$

An immediate consequence of these definitions is that the spectrum of $\mathbf{L}_0$ is σ_0 and the spectrum of $\mathbf{L}_h$ is $\sigma_h = \sigma_-$.

Hypothesis 4.6 (Resolvent estimates)**.** *Assume that there exist positive constants $\omega_0 > 0$, and $c > 0$, such that for all $\omega \in \mathbb{R}$, with $|\omega| \geq \omega_0$, we have that $i\omega$ belongs to the resolvent set of $\mathbf{L}$, and*

$$\|(i\omega \mathbb{I} - \mathbf{L})^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq \frac{c}{|\omega|}, \tag{4.3}$$

While Hypotheses 4.1 and 4.4 are rather easy to check, in applications it is more difficult to check Hypothesis 4.6.

Exercise 4.7. *Prove that Hypothesis 4.6 is satisfied in finite dimensions when $\mathcal{X} = \mathbb{R}^n$.*

[1] If there is no risk of confusion we shall sometimes use the same notation for an operator $\mathbf{L} \in \mathcal{L}(\mathcal{X})$, say, and its restriction to $\mathcal{Z}$, $\mathbf{L}|_{\mathcal{Z}} \in \mathcal{L}(\mathcal{Z})$.

4.2.2 Main Result

In this section we state the center manifold theorem. This result has been proved for the first time in finite dimensions by Pliss [56] in 1964, in the case where the unstable spectrum σ_+ is empty, and by Kelley [40] in 1967, in the case where σ_+ is not empty. There are several versions of these results in infinite dimensions (e.g., see [23], σ_+ is empty, and [53, 71, 43], and the references therein, σ_+ is not empty), and there are analogous results for mappings (e.g., see [47, 51, 35]).

Theorem 4.8 (Center manifold theorem)**.** *Assume that Hypotheses 4.1, 4.4, and 4.6 hold. Then there exists a map* $\mathbf{\Psi} \in \mathcal{C}^k(\mathcal{E}_0, \mathcal{Z}_h)$, *with*

$$\mathbf{\Psi}(0) = 0, \quad D\mathbf{\Psi}(0) = 0, \tag{4.4}$$

and a neighborhood $\mathcal{O}$ *of* 0 *in* $\mathcal{Z}$ *such that the manifold*

$$\mathcal{M}_0 = \{u_0 + \mathbf{\Psi}(u_0)\ ;\ u_0 \in \mathcal{E}_0\} \subset \mathcal{Z} \tag{4.5}$$

has the following properties:

(i) $\mathcal{M}_0$ *is locally invariant, i.e., if* u *is a solution of (4.1) satisfying* $u(0) \in \mathcal{M}_0 \cap \mathcal{O}$ *and* $u(t) \in \mathcal{O}$ *for all* $t \in [0, T]$, *then* $u(t) \in \mathcal{M}_0$ *for all* $t \in [0, T]$.

(ii) The center manifold $\mathcal{M}_0$ *is locally attracting, i.e., any solution of (4.1) that stays in* $\mathcal{O}$ *for all* $t > 0$ *tends exponentially towards a solution of (4.1) on* $\mathcal{M}_0$. *More precisely, if* $u(0) \in \mathcal{O}$ *and the solution* $u(t; u(0))$ *of (4.1) satisfies* $u(t; u(0)) \in \mathcal{O}$ *for all* $t > 0$, *then there exists* $\widetilde{u} \in \mathcal{M}_0 \cap \mathcal{O}$ *and* $\gamma' > 0$ *such that*

$$u(t; u(0)) = u(t; \widetilde{u}) + O(e^{-\gamma' t}) \text{ as } t \to \infty.$$

(Here we denoted by $u(t; u(0))$ *the solution of (4.1) satisfying* $u|_{t=0} = u(0)$*).*

The proof of this theorem and of next theorem 4.15 may be found in [21] Appendices B1 and B5.

Remark 4.9. *Mielke [53] proved theorems 4.8 and 4.15. When spaces* $\mathcal{X}$ *and* $\mathcal{Z}$ *are only Banach spaces, there are various results in the same direction, sometimes needing a little more on* $\mathbf{L}$, *and* $\mathbf{R}$ *(see [21] for the discussion).*

Remark 4.10. *The manifold* $\mathcal{M}_0$ *is called a* local center manifold *of (4.1), and the map* $\mathbf{\Psi}$ *is often referred to as the* reduction function. *Notice that* $\mathcal{M}_0$ *has the same* dimension *as* $\mathcal{E}_0$, *so it is finite-dimensional, and that it is tangent to* $\mathcal{E}_0$ *in* 0, *due to (4.8).*

Center manifolds are fundamental for the study of dynamical systems near "critical situations," and in particular in bifurcation theory. Starting with an infinite-dimensional problem of the form (4.1), the center manifold theorem reduces the study of small solutions, staying sufficiently close to 0, to that of small solutions of a reduced system with finite dimension, equal to the dimension of $\mathcal{E}_0$. Indeed, such solutions belong to the center manifold $\mathcal{M}_0$, and are therefore of the form $u = u_0 + \mathbf{\Psi}(u_0)$. The corollary below shows that solutions on the center manifold are described by a finite-dimensional system of ordinary differential equations, also called *reduced system*, which has the same dimension as $\mathcal{E}_0$.

Corollary 4.11. *Under the assumptions in Theorem 4.8, consider a solution u of (4.1) which belongs to $\mathcal{M}_0$ for $t \in \mathcal{I}$, for some open interval $\mathcal{I} \subset \mathbb{R}$. Then $u = u_0 + \mathbf{\Psi}(u_0)$, and u_0 satisfies*

$$\frac{du_0}{dt} = \mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \mathbf{\Psi}(u_0)). \tag{4.6}$$

Furthermore, the reduction function $\mathbf{\Psi}$ satisfies the equality

$$\begin{aligned} D\mathbf{\Psi}(u_0)\left(\mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \mathbf{\Psi}(u_0))\right) &= \mathbf{L}_h \mathbf{\Psi}(u_0) \\ &\quad + \mathbf{P}_h \mathbf{R}(u_0 + \mathbf{\Psi}(u_0)) \text{ for all } u_0 \in \mathcal{E}_0. \end{aligned} \tag{4.7}$$

Proof. By substituting $u = u_0 + \mathbf{\Psi}(u_0)$ into (4.1) we obtain

$$\frac{du_0}{dt} + D\mathbf{\Psi}(u_0)\frac{du_0}{dt} = \mathbf{L}_0 u_0 + \mathbf{L}_h \mathbf{\Psi}(u_0) + \mathbf{R}(u_0 + \mathbf{\Psi}(u_0)).$$

Projecting this equality with $\mathbf{P}_0$ we find that u_0 satisfies (4.6), and then projecting with $\mathbf{P}_h$ we obtain

$$D\mathbf{\Psi}(u_0)\frac{du_0}{dt} = \mathbf{L}_h \mathbf{\Psi}(u_0) + \mathbf{P}_h \mathbf{R}(u_0 + \mathbf{\Psi}(u_0)).$$

Inserting du_0/dt from (4.6) in the equality above gives (4.7). ∎

Remark 4.12. *In applications it is important to compute the reduced vector field in (4.6), and more precisely its Taylor expansion. Very often it is enough to know the lowest order terms in its Taylor expansion, which can be computed directly from the formula $P_0\mathbf{R}(u_0 + \mathbf{\Psi}(u_0))$. However, there are situations in which we need to know the terms at the next orders. This requires the computation of the Taylor expansion of the reduction function $\mathbf{\Psi}$, as well, which can be done with the help of formula (4.7). We point out that one can compute the Taylor expansions of the reduced vector field and of the reduction function up to the order k, but these computations become more involved as k increases. Several examples of such computations are made in Section 4.4.*

Remark 4.13. *(i) Local center manifolds are in general* not unique *even though the Taylor expansion at the origin is unique. This is due to the occurrence in the proof of the theorem of a smooth cut-off function χ_0 on the space $\mathcal{E}_0$, which is not unique (see [21] Appendix B1). Uniqueness can be achieved under appropriate boundedness conditions on the nonlinearity $\mathbf{R}$: it should be Lipschitzian with sufficiently small Lipschitz constant. We refer to [71, Theorems 1 and 2] for a precise statement of this result. In addition, in this case the resulting center manifold is* global *in the sense that the properties in Theorem 4.8 hold with $\mathcal{O} = \mathcal{Z}$.*

(ii) Center manifolds are in general not analytic *even when the right hand side of the differential equation (4.1) is analytic in u. We refer to [66, 6, 64], and [51, pp. 44–45], [20, p. 126], [70, p. 123] for examples of analytic vector fields leading to nonanalytic center manifolds.*

(iii) A crucial hypothesis in the existing proofs on local center manifolds is Hypothesis 4.4(ii) on the set σ_0, which has to be finite. Without this hypothesis one would expect to construct an infinite- dimensional manifold*. However, this raises a number*

of difficulties, which, so far, have been overcome in only very particular situations [54, 55]. Such a construction would require we first build a "good" projection $\mathbf{P}_0$ associated with the infinite spectral set σ_0, allowing us to obtain a group property for $e^{\mathbf{L}_0 t}$ together with a subexponential growth as $t \to \pm\infty$, and then also to construct a smooth cut-off function χ_0 on the central space $\mathcal{E}_0 = \mathbf{P}_0\mathcal{X}$.

4.2.3 Case when σ_+ is not empty

In many physical examples, we have a situation with a spectrum of the linear operator $\mathbf{L}$ splitted in three parts σ_+, σ_-, and σ_0, where σ_+ is not empty. Then, it is no longer the dynamics as $t \to +\infty$ which interests us, but the existence of bifurcating small solutions for $t \in \mathbb{R}$. The good tool is still a center manifold theorem.

Hypothesis 4.14. *Consider the spectrum σ of the linear operator $\mathbf{L}$, and write*

$$\sigma_+ = \{\lambda \in \sigma \; ; \; Re\lambda > 0\}, \quad \sigma_0 = \{\lambda \in \sigma \; ; \; Re\lambda = 0\}, \quad \sigma_- = \{\lambda \in \sigma \; ; \; Re\lambda < 0\}.$$

We assume that

(i) there exists a positive constant $\gamma > 0$ such that $\sup_{\lambda \in \sigma_-} (Re\lambda) < -\gamma \; , \; \inf_{\lambda \in \sigma_+} (Re\lambda) > \gamma$;

(ii) the set σ_0 consists of a finite number of eigenvalues with finite algebraic multiplicities.

We define the projections $\mathbf{P}_0, \mathbf{P}_h$ and the operators $\mathbf{L}_0, \mathbf{L}_h$, the subspaces $\mathcal{X}_h, \mathcal{Z}_h$ as above. Now, the spectrum of $\mathbf{L}_h$ is $\sigma_h = \sigma_- \cup \sigma_+$.

Theorem 4.15. *Assume that Hypotheses 4.1, 4.14, and 4.6 hold. Then there exists a map $\mathbf{\Psi} \in \mathcal{C}^k(\mathcal{E}_0, \mathcal{Z}_h)$, with*

$$\mathbf{\Psi}(0) = 0, \quad D\mathbf{\Psi}(0) = 0, \tag{4.8}$$

and a neighborhood $\mathcal{O}$ of 0 in $\mathcal{Z}$ such that the manifold

$$\mathcal{M}_0 = \{u_0 + \mathbf{\Psi}(u_0) \; ; \; u_0 \in \mathcal{E}_0\} \subset \mathcal{Z} \tag{4.9}$$

has the following properties:

(i) $\mathcal{M}_0$ is locally invariant, i.e., if u is a solution of (4.1) satisfying $u(0) \in \mathcal{M}_0 \cap \mathcal{O}$ and $u(t) \in \mathcal{O}$ for all $t \in [0, T]$, then $u(t) \in \mathcal{M}_0$ for all $t \in [0, T]$.

(ii) $\mathcal{M}_0$ contains the set of bounded solutions of (4.1) staying in $\mathcal{O}$ for all $t \in \mathbb{R}$, i.e., if u is a solution of (4.1) satisfying $u(t) \in \mathcal{O}$ for all $t \in \mathbb{R}$, then $u(0) \in \mathcal{M}_0$.

This theorem is particularly adapted for reversible systems (see below reversibility symmetry).

4.3 Particular Cases and Extensions

4.3.1 Parameter-Dependent Center Manifolds

In the same frame as above, we consider a *parameter-dependent* differential equation in $\mathcal{X}$ of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u, \mu), \tag{4.10}$$

where $\mathbf{L}$ is a linear operator as in Section 4.2 and $\mathbf{R}$ is defined for (u, μ) in a neighborhood of $(0, 0)$ in $\mathcal{Z} \times \mathbb{R}^m$. Here $\mu \in \mathbb{R}^m$ is a parameter that we assume to be small. More precisely, we keep Hypotheses 4.4, 4.6, and replace Hypothesis 4.1 by the following:

Hypothesis 4.16. *We assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in (4.10) have the following properties:*

(i) $\mathbf{L} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$*;*

(ii) for some $k \geq 2$*, there exist neighborhoods* $\mathcal{V}_u \subset \mathcal{Z}$ *and* $\mathcal{V}_\mu \subset \mathbb{R}^m$ *of* 0 *such that* $\mathbf{R} \in \mathcal{C}^k(\mathcal{V}_u \times \mathcal{V}_\mu, \mathcal{Y})$ *and*

$$\mathbf{R}(0, 0) = 0, \quad D_u\mathbf{R}(0, 0) = 0.$$

Remark 4.17. *The equalities above on* $\mathbf{R}$ *imply that* 0 *is an equilibrium of (4.10) for* $\mu = 0$*, and that* $\mathbf{L}$ *represents the linearization of the vector field about this equilibrium at* $\mu = 0$*. Now, if* $\mathbf{L}$ *has a bounded inverse, then this equilibrium persists for small* μ*. More precisely, by arguing with the implicit function theorem, we find that there is a family of stationary solutions* $u = u(\mu)$ *of (4.10) for* μ *close to* 0*, i.e., such that*

$$\mathbf{L}u(\mu) + \mathbf{R}(u(\mu), \mu) = 0.$$

On the contrary, if $\mathbf{L}$ *does not have a bounded inverse, then this equilibrium may not persist for some values of* μ *near* 0*.*

The analogue of center manifold Theorem 4.8 for the parameter-dependent equation (4.10) is the following result.

Theorem 4.18 (Parameter-dependent center manifolds)**.** *Assume that Hypotheses 4.16, 4.4, and 4.6 hold. Then there exists a map* $\mathbf{\Psi} \in \mathcal{C}^k(\mathcal{E}_0 \times \mathbb{R}^m, \mathcal{Z}_h)$*, with*

$$\mathbf{\Psi}(0, 0) = 0, \quad D_u\mathbf{\Psi}(0, 0) = 0, \tag{4.11}$$

and a neighborhood $\mathcal{O}_u \times \mathcal{O}_\mu$ *of* $(0, 0)$ *in* $\mathcal{Z} \times \mathbb{R}^m$ *such that for* $\mu \in \mathcal{O}_\mu$*, the manifold*

$$\mathcal{M}_0(\mu) = \{u_0 + \mathbf{\Psi}(u_0, \mu) \; ; \; u_0 \in \mathcal{E}_0\} \tag{4.12}$$

has the following properties:

(i) $\mathcal{M}_0(\mu)$ *is locally invariant, i.e., if* u *is a solution of (4.10) satisfying* $u(0) \in \mathcal{M}_0(\mu) \cap \mathcal{O}_u$ *and* $u(t) \in \mathcal{O}_u$ *for all* $t \in [0, T]$*, then* $u(t) \in \mathcal{M}_0(\mu)$ *for all* $t \in [0, T]$*.*

(ii) *The center manifold* $\mathcal{M}_0(\mu)$ *is locally attracting, i.e., any solution of (4.10) that stays in* $\mathcal{O}_u$ *for all* $t > 0$ *tends exponentially towards a solution of (4.10) on* $\mathcal{M}_0(\mu)$*. More precisely, if* $u(0) \in \mathcal{O}_u$ *and the solution* $u(t; u(0))$ *of (4.10) satisfies* $u(t; u(0)) \in \mathcal{O}_u$ *for all* $t > 0$*, then there exists* $\widetilde{u} \in \mathcal{M}_0(\mu) \cap \mathcal{O}_u$ *and* $\gamma' > 0$ *such that*

$$u(t; u(0)) = u(t; \widetilde{u}) + O(e^{-\gamma' t}) \text{ as } t \to \infty.$$

(Here we denoted by $u(t; u(0))$ *the solution of (4.10) satisfying* $u|_{t=0} = u(0)$*.)*

Proof. We consider (4.10) as a particular case of a system of the form (4.1), namely,

$$\frac{d\widetilde{u}}{dt} = \widetilde{\mathbf{L}}\widetilde{u} + \widetilde{\mathbf{R}}(\widetilde{u}), \tag{4.13}$$

by setting

$$\widetilde{u} = (u, \mu),$$

and

$$\begin{aligned} \widetilde{\mathbf{L}}\,\widetilde{u} &= (\mathbf{L}u + D_\mu \mathbf{R}(0,0)\mu, 0), \\ \widetilde{\mathbf{R}}(\widetilde{u}) &= (\mathbf{R}(u,\mu) - D_\mu \mathbf{R}(0,0)\mu, 0). \end{aligned}$$

We show that $\widetilde{\mathbf{L}}$ and $\widetilde{\mathbf{R}}$ verify Hypotheses 4.1, 4.4, and 4.6, with Hilbert spaces

$$\widetilde{\mathcal{X}} = \mathcal{X} \times \mathbb{R}^m, \quad \widetilde{\mathcal{Z}} = \mathcal{Z} \times \mathbb{R}^m,$$

and then the result in the theorem follows from Theorem 4.8.

First, Hypothesis 4.1 is an immediate consequence of Hypothesis 4.16. Next, we show that the spectral sets $\widetilde{\sigma}_-$, $\widetilde{\sigma}_0$ of $\widetilde{\mathbf{L}}$ satisfy

$$\widetilde{\sigma}_- = \sigma_-, \quad \widetilde{\sigma}_0 \setminus \{0\} = \sigma_0 \setminus \{0\}, \tag{4.14}$$

where σ_-, σ_0 are the spectral sets of $\mathbf{L}$, and that $\widetilde{\sigma}_0$ consists of purely imaginary eigenvalues with finite algebraic multiplicities. These properties imply then that Hypothesis 4.4 holds.

Indeed, let us consider the linear equation

$$(\widetilde{\mathbf{L}} - \lambda)\widetilde{u} = \widetilde{v},$$

where $\widetilde{v} = (v, \nu) \in \mathcal{X} \times \mathbb{R}^m$. This means that

$$\begin{aligned} (\mathbf{L} - \lambda)u + D_\mu \mathbf{R}(0,0)\mu &= v, \\ -\lambda\mu &= \nu. \end{aligned}$$

Hence, if $\lambda \neq 0$ we have $\mu = -\nu/\lambda$ and

$$(\mathbf{L} - \lambda)u = v + \lambda^{-1} D_\mu \mathbf{R}(0,0)\nu.$$

Consequently, in $\mathbb{C} \setminus \{0\}$, the resolvent set of $\mathbf{L}$ is identical to the resolvent set of $\widetilde{\mathbf{L}}$. In particular, we have that (4.14) holds. Furthermore, for $\widetilde{\mathbf{L}}$ we can define the spectral projections $\widetilde{\mathbf{P}}_0$, $\widetilde{\mathbf{P}}_h$, and the corresponding spectral spaces $\widetilde{\mathcal{E}}_0$, $\widetilde{\mathcal{X}}_h$ as in Section 4.2.1.

Next, notice that $\mathcal{X}_h \times \{0\}$ is an invariant subspace for $\widetilde{\mathbf{L}}$, since

$$\widetilde{\mathbf{L}}(u_h, 0) = (\mathbf{L}_h u_h, 0) \in \mathcal{X}_h \times \{0\} \text{ for all } u_h \in \mathcal{Z}_h.$$

From this equality we further deduce that

$$\sigma(\widetilde{\mathbf{L}}\big|_{\mathcal{X}_h \times \{0\}}) = \sigma(\mathbf{L}_h) = \sigma_- = \widetilde{\sigma}_-.$$

Consequently, $\mathcal{X}_h \times \{0\} \subset \widetilde{\mathcal{X}}_h$, and since

$$\text{codim}\widetilde{\mathcal{X}}_h \leq \text{codim}\,(\mathcal{X}_h \times \{0\}) = \dim \mathcal{E}_0 + m < \infty,$$

we conclude that

$$\dim \widetilde{\mathcal{E}}_0 = \text{codim}\widetilde{\mathcal{X}}_h < \infty.$$

In particular, this shows that $\widetilde{\sigma}_0$ consists of purely imaginary eigenvalues with finite algebraic multiplicities and proves Hypothesis 4.4.

In order to prove Hypothesis 4.6 it is enough to show that $\widetilde{\mathcal{X}}_h = \mathcal{X}_h \times \{0\}$, and then the conditions on $\widetilde{\mathbf{L}}$ in Hypothesis 4.6 follow from the analogue ones on $\mathbf{L}$. We claim that

$$\widetilde{\mathcal{E}}_0 = \{(u_0 - \mathbf{L}_h^{-1} D_\mu \mathbf{R}_h(0,0)\mu, \mu)\ ;\ u_0 \in \mathcal{E}_0,\ \mu \in \mathbb{R}^m\} =: \mathcal{F}_0.$$

Then this implies that

$$\text{codim}\widetilde{\mathcal{X}}_h = \dim \widetilde{\mathcal{E}}_0 = \dim \mathcal{E}_0 + m = \text{codim}\,(\mathcal{X}_h \times \{0\}),$$

and since $\mathcal{X}_h \times \{0\} \subset \widetilde{\mathcal{X}}_h$ we conclude that $\widetilde{\mathcal{X}}_h = \mathcal{X}_h \times \{0\}$.

It remains to prove the claim $\widetilde{\mathcal{E}}_0 = \mathcal{F}_0$. First, take $\widetilde{u} = (u, \mu) \in \widetilde{\mathcal{E}}_0 \subset \widetilde{\mathcal{Z}}$. We write $u = u_0 + u_h$ with $u_0 \in \mathcal{E}_0$, $u_h \in \mathcal{Z}_h$, and compute

$$\widetilde{\mathbf{L}}\widetilde{u} = (\mathbf{L}_h u_h + D_\mu \mathbf{R}_h(0,0)\mu, 0) + (\mathbf{L}_0 u_0 + D_\mu \mathbf{R}_0(0,0)\mu, 0),$$

where $\mathbf{R}_h = \mathbf{P}_h \mathbf{R}$ and $\mathbf{R}_0 = \mathbf{P}_0 \mathbf{R}$. The first term on the right hand side of the above equality belongs to $\mathcal{X}_h \times \{0\} \subset \widetilde{\mathcal{X}}_h$, whereas the second term belongs to $\mathcal{E}_0 \times \{0\} \subset \widetilde{\mathcal{E}}_0$. Then, since $\widetilde{\mathbf{L}}\widetilde{u} \in \widetilde{\mathcal{E}}_0$, the first term vanishes, so that

$$\mathbf{L}_h u_h + D_\mu \mathbf{R}_h(0,0)\mu = 0.$$

Now $\mathbf{L}_h$ has a bounded inverse because 0 does not belong to its spectrum, so that we find

$$u_h = -\mathbf{L}_h^{-1} D_\mu \mathbf{R}_h(0,0)\mu.$$

Summarizing, for $\widetilde{u} \in \widetilde{\mathcal{E}}_0$, we have

$$\widetilde{u} = (u, \mu) = (u_0 + u_h, \mu) = (u_0 - \mathbf{L}_h^{-1} D_\mu \mathbf{R}_h(0,0)\mu, \mu),$$

which proves that $\widetilde{\mathcal{E}}_0 \subset \mathcal{F}_0$.

Next, notice that

$$\widetilde{\mathbf{L}}(u_0 - \mathbf{L}_h^{-1} D_\mu \mathbf{R}_h(0,0)\mu, \mu) = (\mathbf{L}_0 u_0 + D_\mu \mathbf{R}_0(0,0)\mu, 0) \in \mathcal{E}_0 \times \{0\} \subset \mathcal{F}_0,$$

so that $\mathcal{F}_0$ is an invariant subspace for $\widetilde{\mathbf{L}}$. Consider the bases $\{e_j; j = 1, \dots, \dim \mathcal{E}_0\}$ and $\{f_k; k = 1, \dots, m\}$ of $\mathcal{E}_0$ and $\mathbb{R}^m$, respectively. Then the set

$$\{(e_j, 0), (-\mathbf{L}_h^{-1} D_\mu \mathbf{R}_h(0,0) f_k, f_k) \; ; \; j = 1, \dots, \dim \mathcal{E}_0, \; k = 1, \dots, m\}$$

is a basis for $\mathcal{F}_0$, in which we find that the matrix of $\widetilde{\mathbf{L}}\big|_{\mathcal{F}_0}$ is of the form

$$\begin{pmatrix} M_0 & M_1 \\ 0 & 0 \end{pmatrix},$$

with M_0 the matrix of $\mathbf{L}_0$ in the basis $\{e_j; j = 1, \dots, \dim \mathcal{E}_0\}$ and M_1 a matrix of size $m \times \dim \mathcal{E}_0$. The set of eigenvalues of M_0 is precisely the set σ_0, and we then conclude that

$$\sigma(\widetilde{\mathbf{L}}\big|_{\mathcal{F}_0}) = \sigma_0 \cup \{0\} \subset \widetilde{\sigma}_0.$$

In particular, this implies that $\mathcal{F}_0 \subset \widetilde{\mathcal{E}}_0$, which completes the proof of $\widetilde{\mathcal{E}}_0 = \mathcal{F}_0$. ∎

Remark 4.19. *The analogue of the* reduced equation *(4.6) in this situation is*

$$\frac{du_0}{dt} = \mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \mathbf{\Psi}(u_0, \mu), \mu) \stackrel{def}{=} f(u_0, \mu), \tag{4.15}$$

where we observe that $f(0,0) = 0$ *and* $D_{u_0} f(0,0) = \mathbf{L}_0$ *has the spectrum* σ_0*. Similarly, we have the analogue of the equality (4.7),*

$$\begin{aligned} D_{u_0} \mathbf{\Psi}(u_0, \mu) f(u_0, \mu) \;\; = \;\; & \mathbf{L}_h \mathbf{\Psi}(u_0, \mu) \\ & + \mathbf{P}_h \mathbf{R}(u_0 + \mathbf{\Psi}(u_0, \mu), \mu) \text{ for all } u_0 \in \mathcal{E}_0. \end{aligned} \tag{4.16}$$

Exercise 4.20. *Consider a system of the form (4.10) for which* 0 *is a solution for all values of* μ*, i.e., such that* $\mathbf{R}(0, \mu) = 0$ *for all* μ *in a neighborhood of* 0 *in* $\mathbb{R}^m$*. Show that*

$$\mathbf{\Psi}(0, \mu) = 0, \quad f(0, \mu) = 0,$$

for μ *sufficiently small. Furthermore, set*

$$\mathbf{L}_\mu = \mathbf{L} + D_u \mathbf{R}(0, \mu) \in \mathcal{L}(\mathcal{Z}, \mathcal{X}) \quad \text{and} \quad \mathbf{A}_\mu = \frac{\partial f}{\partial u_0}(0, \mu).$$

Show that eigenvalues of $\mathbf{A}_\mu$ *are precisely the eigenvalues of* $\mathbf{L}_\mu$*, which are the continuation for small* μ *of the purely imaginary eigenvalues of* $\mathbf{L}$ *(i.e., those of* $\mathbf{L}_0$*)).*
Hint: *Identify the terms linear in* u_0 *in the identity*

$$(\mathbb{I} + D_{u_0} \mathbf{\Psi}(u_0, \mu))\, f(u_0, \mu) = \mathbf{L}(u_0 + \mathbf{\Psi}(u_0, \mu)) + \mathbf{R}(u_0 + \mathbf{\Psi}(u_0, \mu), \mu) \text{ for all } u_0 \in \mathcal{E}_0.$$

Remark 4.21 (Case when $\boldsymbol{\sigma}_0$ does not lie on the imaginary axis)**.** *A situation arising in some applications is one in which the eigenvalues in* σ_0 *of the operator* $\mathbf{L}$ *in (4.10) do not lie on the imaginary axis but stay close to the imaginary axis. More precisely, we still have the spectral decomposition in Hypothesis 4.4, satisfying the properties (i) and (ii), but with* σ_0 *such that*

$$\sigma_0 = \{\lambda \in \sigma \; ; \; |Re \lambda| \leq \delta\} \tag{4.17}$$

for some $\delta \ll \gamma$ sufficiently small. This means that σ_0 consists of a finite number of eigenvalues λ_j, $j = 1, \ldots, r$ of $\mathbf{L}$, with real parts that are small but not necessarily 0*:*

$$Re\lambda_j = \varepsilon_j, \quad |\varepsilon_j| \leq \delta, \quad j = 1, \ldots, r.$$

In such a situation we can apply the result in Theorem 4.18 by arguing in the following way:

Consider the bounded linear operator

$$\mathbf{A}_\nu = \sum_{j=1}^{r} \nu_j \mathbf{P}_j \text{ for } \nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r,$$

where $\mathbf{P}_j$ denotes the spectral projection associated with the eigenvalue $\lambda_j \in \sigma_0$ of $\mathbf{L}$. When $\nu = \varepsilon$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$, the operator

$$\mathbf{L}' = \mathbf{L} - \mathbf{A}_\varepsilon, \quad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_r),$$

satisfies Hypothesis 4.4, the effect of adding $-\mathbf{A}_\varepsilon$ to $\mathbf{L}$ being that all eigenvalues in σ_0 are shifted on the imaginary axis. Consequently, we can apply the result in Theorem 4.18 to the modified system

$$\frac{du}{dt} = \mathbf{L}'u + \mathbf{R}'(u, \mu'),$$

where $\mu' = (\mu, \nu)$ and

$$\mathbf{R}'(u, \mu') = \mathbf{A}_\nu u + \mathbf{R}(u, \mu),$$

which satisfies the hypotheses in Theorem 4.18 with the parameter $\mu' = (\mu, \nu) \in \mathbb{R}^{m+r}$. We recover the original equation by taking $\nu = \varepsilon$, and find the invariant manifolds $\mathcal{M}_0(\mu, \varepsilon)$ for this equation, provided ε is sufficiently small, such that $(0, \varepsilon)$ belongs to the neighborhood $\mathcal{O}_{\mu'}$ of $(0, 0)$ in $\mathbb{R}^{m+r}$ given by Theorem 4.18. This latter property is achieved when δ in (4.17) is sufficiently small, i.e., when the eigenvalues in σ_0 are close enough to the imaginary axis.

Remark 4.22. *(i) In (4.10) the parameter μ occurs only in the term $\mathbf{R}$, which takes values in $\mathcal{Y}$. A more general study would be for cases where μ also occurs in the linear terms which take values in $\mathcal{X}$. Then one would have a family of operators $\mathbf{L}_\mu$ with domains which may also depend upon μ. Such a situation requires a more delicate analysis, which does not enter in our setting.*

(ii) It is possible to develop the theory for a parameter μ lying in a (infinite-dimensional) Banach space instead of $\mathbb{R}^m$. Nevertheless, for such a situation one needs to go back and adapt the proof of the general result in Theorem 4.8. The proof of Theorem 4.18 given above does not extend to this situation, since it relies upon the fact that $\mathbb{R}^m$ is finite-dimensional (one has that $\dim \widetilde{\mathcal{E}}_0 = \dim \mathcal{E}_0 + m$, and this quantity is infinite when $\mathbb{R}^m$ is replaced by an infinite- dimensional Banach space, so that the extended system (4.13) does not satisfy Hypothesis 4.4(ii)). We refer the reader to [36] for an example of a problem with a parameter varying in a function space, and for which the continuity of the reduction function $\mathbf{\Psi}$ with respect to the parameter, is only valid in $\mathcal{X}$, not in $\mathcal{Z}$.

4.3.2 Nonautonomous Center Manifolds

We present in this section an extension of the result of center manifold Theorem 4.8 to the case of nonautonomous equations of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u,t). \tag{4.18}$$

We replace here Hypothesis 4.1 by the following assumptions on $\mathbf{L}$ and $\mathbf{R}$.

Hypothesis 4.23. *We assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in (4.18) have the following properties:*

(i) $\mathbf{L} \in \mathcal{L}(\mathcal{Z},\mathcal{X})$;

(ii) for some $k \geq 2$, *there exists a neighborhood* $\mathcal{V} \subset \mathcal{Z}$ *of* 0 *such that* $\mathbf{R} \in \mathcal{C}^k(\mathcal{V}\times\mathbb{R},\mathcal{Y})$ *and*

$$\mathbf{R}(0,t) = 0, \quad D_u\mathbf{R}(0,t) = 0.$$

In addition, we assume that for any sufficiently small ε, *there exist positive constants* $\delta_0(\varepsilon) = O(\varepsilon^2)$ *and* $\delta_1(\varepsilon) = O(\varepsilon)$ *such that*

$$\sup_{u\in B_\varepsilon(\mathcal{Z})} \|\mathbf{R}(u,t)\|_{\mathcal{Y}} = \delta_0(\varepsilon), \qquad \sup_{u\in B_\varepsilon(\mathcal{Z})} \|D_u\mathbf{R}(u,t)\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} = \delta_1(\varepsilon). \tag{4.19}$$

The equalities in the formula (4.19) above, show that the nonlinear term $\mathbf{R}$ is bounded with respect to all $t \in \mathbb{R}$, uniformly for u in any sufficiently small closed ball $B_\varepsilon(\mathcal{Z})$. Furthermore, the dependency in t of the system (4.18) is in the nonlinear term $\mathbf{R}$, only. In this sense, the following theorem is a "perturbation" result of center manifold Theorem 4.8.

Theorem 4.24 (Nonautonomous center manifolds)**.** *Assume that Hypotheses 4.23, 4.4, and 4.6 hold. Then, there exist a map* $\mathbf{\Psi} \in \mathcal{C}^k(\mathcal{E}_0\times\mathbb{R},\mathcal{Z}_h)$ *and* $c > 0$, *with*

$$\mathbf{\Psi}(0,t) = 0, \quad D_{u_0}\mathbf{\Psi}(0,t) = 0,$$

and

$$\sup_{u_0\in B_\varepsilon(\mathcal{E}_0)} \|\mathbf{\Psi}(u_0,t)\|_{\mathcal{Z}} = c\delta_0(\varepsilon), \qquad \sup_{u_0\in B_\varepsilon(\mathcal{E}_0)} \|D_u\mathbf{\Psi}(u_0,t)\|_{\mathcal{L}(\mathcal{Z})} = c\delta_1(\varepsilon),$$

for sufficiently small ε, *and a neighborhood* $\mathcal{O}$ *of* 0 *in* $\mathcal{Z}$ *such that the manifold*

$$\mathcal{M}_0(t) = \{u_0 + \mathbf{\Psi}(u_0,t)\ ;\ (u_0,t) \in B_\varepsilon(\mathcal{E}_0)\times\mathbb{R}\} \subset \mathcal{Z}$$

has the following properties:

(i) the set $\{(t,u(t)) \in \mathbb{R}\times\mathcal{M}_0(t)\}$ *is a local integral manifold of (4.18);*

(ii) the manifold $\mathcal{M}_0(t)$ *is locally attracting. Any solution* u *of (4.18) staying in* $\mathcal{O}$ *for all* $t \in \mathbb{R}$ *tends exponentially toward a solution lying on the set* $(t,u(t)) \in \mathbb{R}\times\mathcal{M}_0(t)$.

We give a brief proof of this result in Appendix B3 of [21] (see also [52] for a complete proof).

Remark 4.25. *The analogue of the reduced equation (4.6) in this situation is*

$$\frac{du_0}{dt} = \mathbf{L}_0 u_0 + \mathbf{P}_0 \mathbf{R}(u_0 + \mathbf{\Psi}(u_0,t),t) \stackrel{def}{=} f(u_0,t), \tag{4.20}$$

whereas the analogue of the equality (4.7) is

$$\begin{aligned} \partial_t \mathbf{\Psi}(u_0,t) + D_{u_0}\mathbf{\Psi}(u_0,t) f(u_0,t) &= \mathbf{L}_h \mathbf{\Psi}(u_0,t) \\ &+ \mathbf{P}_h \mathbf{R}(u_0 + \mathbf{\Psi}(u_0,t),t) \text{ for all } u_0 \in \mathcal{E}_0. \end{aligned}$$

There are at least two particular cases of equation (4.18) that are important in applications:

(i) the case in which the map $\mathbf{R}$ is periodic with respect to t, and

(ii) the case in which $\lim_{t\to\infty} \mathbf{R}(u,t) \to \mathbf{R}_\infty(u)$ or $\lim_{t\to-\infty} \mathbf{R}(u,t) \to \mathbf{R}_{-\infty}(u)$.

In these cases the reduction function $\mathbf{\Psi}$, and then also the reduced system, has similar properties. The following result is proved in [21] Appendix B3:

Corollary 4.26 (Special cases)**.** *Assume that the hypothesis in Theorem 4.24 holds.*

(i) If the map $\mathbf{R}$ is periodic with respect to t, $\mathbf{R}(u,t) = \mathbf{R}(u,t+\tau)$ for some $\tau > 0$, then one can find a reduction function $\mathbf{\Psi}$ that is periodic, with the same period, namely $\mathbf{\Psi}(u_0,t) = \mathbf{\Psi}(u_0,t+\tau)$ for any $(u_0,t) \in B_\varepsilon(\mathcal{E}_0) \times \mathbb{R}$.

(ii) Assume that there exist a map $\mathbf{R}_\infty \in \mathcal{C}^k(\mathcal{V},\mathcal{X})$ and $d_0 > 0$ such that

$$\|\mathbf{R}(u,t) - \mathbf{R}_\infty(u)\|_{\mathcal{X}} \leq ce^{-d_0 t} \text{ for all } (u,t) \in \mathcal{V} \times \mathbb{R}^+.$$

Then the result in center manifold Theorem 4.8 holds for the autonomous equation

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}_\infty(u), \tag{4.21}$$

and there exists $c' > 0$ such that

$$\|\mathbf{\Psi}(u_0,t) - \mathbf{\Psi}_\infty(u_0)\|_{\mathcal{Z}_h} \leq c' e^{-d_0 t} \text{ for all } (u_0,t) \in B_\varepsilon(\mathcal{E}_0) \times \mathbb{R}^+,$$

where $\mathbf{\Psi}_\infty$ is the reduction function for the autonomous equation (4.21). A similar result holds when $\|\mathbf{R}(u,t) - \mathbf{R}_{-\infty}(u)\|_{\mathcal{X}} \leq ce^{d_0 t}$ for all $(u,t) \in \mathcal{V} \times \mathbb{R}^-$.

4.3.3 Parameter dependent Center manifold in the case σ_+ not empty

As in the non parameter dependent case, we have the following

Theorem 4.27 (Parameter-dependent center manifolds in the case σ_+ not empty)**.** *Assume that Hypotheses 4.16, 4.14, and 4.6 hold. Then there exists a map $\mathbf{\Psi} \in \mathcal{C}^k(\mathcal{E}_0 \times \mathbb{R}^m, \mathcal{Z}_h)$, with*

$$\mathbf{\Psi}(0,0) = 0, \quad D_u \mathbf{\Psi}(0,0) = 0, \tag{4.22}$$

and a neighborhood $\mathcal{O}_u \times \mathcal{O}_\mu$ of $(0,0)$ in $\mathcal{Z} \times \mathbb{R}^m$ such that for $\mu \in \mathcal{O}_\mu$, the manifold

$$\mathcal{M}_0(\mu) = \{u_0 + \mathbf{\Psi}(u_0,\mu) \; ; \; u_0 \in \mathcal{E}_0\} \tag{4.23}$$

has the following properties:

(i) $\mathcal{M}_0(\mu)$ *is locally invariant, i.e., if* u *is a solution of (4.10) satisfying* $u(0) \in \mathcal{M}_0(\mu) \cap \mathcal{O}_u$ *and* $u(t) \in \mathcal{O}_u$ *for all* $t \in [0, T]$*, then* $u(t) \in \mathcal{M}_0(\mu)$ *for all* $t \in [0, T]$.

(ii) $\mathcal{M}_0(\mu)$ *contains the set of bounded solutions of (4.10) staying in* $\mathcal{O}_u$ *for all* $t \in \mathbb{R}$*, i.e., if* u *is a solution of (4.10) satisfying* $u(t) \in \mathcal{O}_u$ *for all* $t \in \mathbb{R}$*, then* $u(0) \in \mathcal{M}_0(\mu)$.

4.3.4 Symmetries

We discuss in this section three cases of equations possessing a certain symmetry. In each case we show that this symmetry is inherited by both the reduction function $\mathbf{\Psi}$ and the reduced system.

Equivariant Systems We start with the case of an equation that is equivariant under the action of a linear operator. More precisely, we make the following assumptions.

Hypothesis 4.28 (Equivariant equation)**.** *We assume that there exists a linear operator* $\mathbf{T} \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Z})$*, which commutes with the vector field in equation (4.1),*

$$\mathbf{TL}u = \mathbf{LT}u, \quad \mathbf{TR}(u) = \mathbf{R}(\mathbf{T}u).$$

We further assume that the restriction $\mathbf{T}_0$ *of* $\mathbf{T}$ *to the subspace* $\mathcal{E}_0$ *is an isometry.*

Notice that the fact that the operator $\mathbf{T}$ commutes with the vector field in the equation (4.1) implies that the subspace $\mathcal{E}_0$ is invariant under the action of $\mathbf{T}$, so that the restriction $\mathbf{T}_0$ in the hypothesis above is well defined. Indeed, since $\mathbf{T}$ commutes with $\mathbf{L}$, it also commutes with its resolvent $(\lambda\mathbb{I} - \mathbf{L})^{-1}$, and from the Dunford integral formula (4.2) it follows that $\mathbf{T}$ commutes with the spectral projector $\mathbf{P}_0$. Consequently, the spectral subspace $\mathcal{E}_0$ associated with $\mathbf{P}_0$ is invariant under the action of $\mathbf{T}$.

We show in [21] Appendix B4 that the following result holds in this situation.

Theorem 4.29 (Center manifold theorem for equivariant equations)**.** *Under the assumptions in Theorem 4.8, we further assume that Hypothesis 4.28 holds. Then one can find a reduction function* $\mathbf{\Psi}$ *in Theorem 4.8 which commutes with* $\mathbf{T}$*, i.e.,*

$$\mathbf{T\Psi}(u_0) = \mathbf{\Psi}(\mathbf{T}_0 u_0) \text{ for all } u_0 \in \mathcal{E}_0,$$

and such that the vector field in the reduced equation (4.6) commutes with $\mathbf{T}_0$.

We point out that analogous results hold for the parameter-dependent equation (4.10) and in the nonautonomous case for the equation (4.18).

Reversible Systems Next, we consider the case of reversible equations, when the vector field in (4.1) anticommutes with a symmetry $\mathbf{S}$. More precisely, we make the following assumptions.

Hypothesis 4.30 (Reversible equation)**.** *Assume that there exists a linear symmetry* $\mathbf{S} \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Z})$*, with*

$$\mathbf{S}^2 = \mathbb{I}, \quad \mathbf{S} \neq \mathbb{I},$$

and which anticommutes with the vector field in (4.1),

$$\mathbf{SL}u = -\mathbf{LS}u, \quad \mathbf{SR}(u) = -\mathbf{R}(\mathbf{S}u). \tag{4.24}$$

Notice that in this case, if $t \mapsto u(t)$ is a solution of (4.1), then $t \mapsto \mathbf{S}u(-t)$ is also a solution of (4.1). Moreover, the spectrum of the linear operator $\mathbf{L}$ is symmetric with respect to the origin in the complex plane. Indeed, from the first equality in (4.24) we deduce that

$$\mathbf{S}(\lambda\mathbb{I} - \mathbf{L})^{-1} = (\lambda\mathbb{I} + \mathbf{L})^{-1}\mathbf{S},$$

which shows that the resolvent set $\rho(\mathbf{L})$ as well as its complement $\sigma(\mathbf{L})$ are symmetric with respect to the origin. In particular, for real systems, besides the usual symmetry with respect to the real axis, in this case the spectrum of $\mathbf{L}$ is also symmetric with respect to the imaginary axis. We also point out that if λ is an eigenvalue of $\mathbf{L}$ with the associated eigenvector ζ, then $-\lambda$ is an eigenvalue with the associated eigenvector $\mathbf{S}\zeta$.

As in the case of equivariant equations with Hypothesis 4.28, we have that the spectral subspace $\mathcal{E}_0$ is invariant under the action of $\mathbf{S}$. Indeed, since the spectrum of the operator $\mathbf{L}$ is symmetric with respect to the origin in the complex plane, we may choose the curve Γ in the Dunford integral formula (4.2) such that it is also symmetric with respect to the origin in the complex plane. Then a direct calculation shows that the spectral projection $\mathbf{P}_0$ given by (4.2) commutes with $\mathbf{S}$, so that $\mathcal{E}_0$ is invariant under the action of $\mathbf{S}$.

By arguing as in the case of equivariant equations, we obtain here the following result.

Theorem 4.31 (Center manifold theorem for reversible equations)**.** *Under the assumptions of Theorem 4.8, we further assume that Hypothesis 4.30 holds. Then one can find a reduction function $\mathbf{\Psi}$ in Theorem 4.8 that commutes with $\mathbf{S}$,*

$$\mathbf{S\Psi}(u_0) = \mathbf{\Psi}(\mathbf{S}_0 u_0) \text{ for all } u_0 \in \mathcal{E}_0,$$

where $\mathbf{S}_0$ is the restriction of $\mathbf{S}$ to the subspace $\mathcal{E}_0$ and such that the reduced equation is reversible, i.e., the vector field in (4.6) anticommutes with $\mathbf{S}_0$.

A similar result holds for the parameter-dependent equation (4.10).

Continuous Symmetry We end this section with the case where equation (4.1) is equivariant under a one-parameter group of isometries. We focus on the case of the underlying group $\mathbb{R}$, and, instead of a single equilibrium at the origin, the equation has a "line" of equilibria. This situation is encountered in the applications in Subsections 6.2, 6.3. Other groups of symmetries can be treated in the same spirit, however, this may require more specific tools and further evolved algebra. We refer the reader to the book [9] for such cases. More precisely, we make here the following hypotheses.

Hypothesis 4.32 (Continuous symmetry)**.** *Assume that there exists a continuous one-parameter group of isometries $(\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}} \subset \mathcal{L}(\mathcal{Z}) \cap \mathcal{L}(\mathcal{X})$, which commutes with the vector field in (4.1), that is, such that the following properties hold:*

(i) the map $\alpha \in \mathbb{R} \mapsto \mathbf{T}_\alpha \in \mathcal{L}(\mathcal{Z}) \cap \mathcal{L}(\mathcal{X})$ is continuous;

(ii) $\mathbf{T}_0 = \mathbb{I}$ and $\mathbf{T}_{\alpha+\beta} = \mathbf{T}_\alpha\mathbf{T}_\beta$ for all $\alpha,\ \beta \in \mathbb{R}$;

(iii) $\mathbf{T}_\alpha\mathbf{L}u = \mathbf{L}\mathbf{T}_\alpha u$ and $\mathbf{T}_\alpha\mathbf{R}(u) = \mathbf{R}(\mathbf{T}_\alpha u)$ for all $\alpha \in \mathbb{R}$.

Further assume that the infinitesimal generator $\boldsymbol{\tau}$ *of the group* $(\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}} \subset \mathcal{L}(\mathcal{X})$ *belongs to* $\mathcal{L}(\mathcal{Z},\mathcal{X})$*,*

$$\boldsymbol{\tau} := \frac{d\mathbf{T}_\alpha}{d\alpha}\Big|_{\alpha=0} \in \mathcal{L}(\mathcal{Z},\mathcal{X}).$$

Hypothesis 4.33 (Equilibria)**.** *Assume that equation (4.1) has a nontrivial equilibrium* $u^* \in \mathcal{Z}$*,*

$$\mathbf{L}u^* + \mathbf{R}(u^*) = 0, \quad u^* \neq 0,$$

satisfying $\boldsymbol{\tau}u^* \in \mathcal{Z} \setminus \{0\}$*.*

An immediate consequence of the hypotheses above is that equation (4.1) possesses a *line of equilibria* given by $\{\mathbf{T}_\alpha u^* \in \mathcal{Z}; \alpha \in \mathbb{R}\}$. Furthermore, since $\boldsymbol{\tau}u^* \in \mathcal{Z}$, we may differentiate the identity

$$\mathbf{L}\mathbf{T}_\alpha u^* + \mathbf{R}(\mathbf{T}_\alpha u^*) = 0$$

at $\alpha = 0$ and obtain

$$\mathbf{L}\boldsymbol{\tau}u^* + D\mathbf{R}(u^*)\boldsymbol{\tau}u^* = 0. \tag{4.25}$$

This shows that $\boldsymbol{\tau}u^*$ belongs to the kernel of the linearization $\mathbf{L}+D\mathbf{R}(u^*)$ of the vector field at the equilibrium u^* (this eigenvector is often called the "Goldstone mode" by physicists).

Our purpose is to construct *a local center manifold along this line of equilibria* in $\mathcal{Z}$, taking into account the continuous symmetry of the equation. We make the Ansatz

$$u(t) = \mathbf{T}_{\alpha(t)}(u^* + v(t)), \tag{4.26}$$

replacing the unknown u by the pair (α, v), with $\alpha(t) \in \mathbb{R}$ and $v(t) \in \mathcal{Z}$ satisfying a transversality condition that we define now. For this we decompose the space $\mathcal{X}$ in the subspace spanned by $\boldsymbol{\tau}u^*$, parallel to the line of equilibria, and a complementary subspace. Consider the linear form φ^* in the dual space $\mathcal{X}^*$ such that $\langle \boldsymbol{\tau}u^*, \varphi^*\rangle = 1$ (e.g., see [39, p. 135]). We define the subspace $\mathcal{H} \subset \mathcal{X}$ transverse to $\boldsymbol{\tau}u^*$,

$$\mathcal{H} = \{v \in \mathcal{X} \; ; \; \langle v, \varphi^*\rangle = 0\},$$

which provides us with a decomposition of $\mathcal{X}$ into two complementary closed subspaces,

$$\mathcal{X} = \{\boldsymbol{\tau}u^*\} \oplus \mathcal{H}.$$

The linear operators

$$\mathbf{\Pi}_0 u = \langle u, \varphi^*\rangle \boldsymbol{\tau}u^*, \quad \mathbf{\Pi}_{\mathcal{H}} = \mathbb{I} - \mathbf{\Pi}_0$$

are projections onto the subspaces $\{\boldsymbol{\tau}u^*\}$ and $\mathcal{H}$, respectively. Since $\boldsymbol{\tau}u^* \in \mathcal{Z}$, we have that $\Pi_{\mathcal{H}}u \in \mathcal{Z}$ if $u \in \mathcal{Z}$, so that we have a similar decomposition for $\mathcal{Z}$. We now choose v in (4.26) such that $v(t)$ belongs to $\mathcal{H}$, i.e.,

$$\mathbf{\Pi}_0 v(t) = 0 \quad \Longleftrightarrow \quad \langle v(t), \varphi^*\rangle = 0.$$

Next, we substitute the Ansatz (4.26) into the equation (4.1) and obtain the equation

$$\boldsymbol{\tau}\mathbf{T}_\alpha(u^* + v)\frac{d\alpha}{dt} + \mathbf{T}_\alpha\frac{dv}{dt} = \mathbf{L}\mathbf{T}_\alpha v + \mathbf{R}(\mathbf{T}_\alpha(u^* + v)) - \mathbf{R}(\mathbf{T}_\alpha u^*),$$

where we have used the fact that $\mathbf{T}_\alpha u^*$ is an equilibrium of (4.1). Using the equivariance property in Hypothesis 4.32(iii) we find

$$(\boldsymbol{\tau} u^* + \boldsymbol{\tau} v)\frac{d\alpha}{dt} + \frac{dv}{dt} = \mathbf{A}v + \tilde{\mathbf{R}}(v),$$

in which

$$\mathbf{A}v = \mathbf{L}v + D\mathbf{R}(u^*)v, \quad \tilde{\mathbf{R}}(v) = \mathbf{R}(u^* + v) - \mathbf{R}(u^*) - D\mathbf{R}(u^*)v.$$

Projecting successively with $\mathbf{\Pi}_0$ and $\mathbf{\Pi}_\mathcal{H}$, this gives the first order system for (α, v),

$$\frac{d\alpha}{dt} = (1 + \langle \boldsymbol{\tau} v, \varphi^* \rangle)^{-1} \langle \mathbf{A}v + \tilde{\mathbf{R}}(v), \varphi^* \rangle \stackrel{\text{def}}{=} g(v) \tag{4.27}$$

$$\frac{dv}{dt} = \mathbf{\Pi}_\mathcal{H}\mathbf{A}v + \mathbf{\Pi}_\mathcal{H}\tilde{\mathbf{R}}(v) - g(v)\mathbf{\Pi}_\mathcal{H}\boldsymbol{\tau} v, \tag{4.28}$$

which holds for $v \in \mathcal{Z}$ sufficiently small.

The key property of the system (4.27)–(4.28) is that the vector field is independent of α, which in particular does not appear in the equation (4.28). This equation decouples, so that we can solve it separately, and once v is known we obtain α from the first equation. The differential equation (4.28) is of the form of (4.1), with the spaces $\mathcal{X}$, $\mathcal{Z}$, replaced by

$$\mathcal{X}' = \mathcal{H}, \quad \mathcal{Z}' = \mathbf{\Pi}_\mathcal{H}\mathcal{Z},$$

respectively, and operators $\mathbf{L}$ and $\mathbf{R}$ replaced by

$$\mathbf{L}' = \mathbf{\Pi}_\mathcal{H}\mathbf{A}, \quad \mathbf{R}'(v) = \mathbf{\Pi}_\mathcal{H}(\tilde{\mathbf{R}}(v) - g(v)\boldsymbol{\tau} v), \tag{4.29}$$

respectively. In particular, this means that thanks to the choice of the Ansatz (4.26), the dimension of the problem is decreased by one, the space $\mathcal{X}$ being replaced by $\mathcal{H}$. In fact we suppressed the direction τu^*, which belongs to the kernel of $\mathbf{A}$ as shown by (4.25). Furthermore, once we obtain a local center manifold for equation (4.28), we have a center manifold for equation (4.1), with one additional dimension, in a neighborhood of the line of stationary solutions $\{\mathbf{T}_\alpha u^* \in \mathcal{Z}; \alpha \in \mathbb{R}\}$. More precisely, we have the following result.

Theorem 4.34 (Center manifolds in presence of continuous symmetry)**.** *Assume that Hypothesis 4.1 holds and that the linear operator $\mathbf{L}' = \mathbf{\Pi}_\mathcal{H}\mathbf{A}$ in (4.29) acting in $\mathcal{X}'$ satisfies Hypotheses 4.4 and 4.6. Then for the differential equation (4.28) the result in Theorem 4.8 holds.*

Let $\mathcal{O}'$, $\mathbf{\Psi}'$, and $\mathcal{E}_0'$ be respectively the neighborhood of the origin in $\mathcal{Z}'$, the reduction function, and the spectral subspace, given by Theorem 4.8 for (4.28). Consider the "tubular" neighborhood

$$\mathcal{O} = \{\mathbf{T}_\alpha(u^* + v) \; ; \; v \in \mathcal{O}', \; \alpha \in \mathbb{R}\} \subset \mathcal{Z}$$

of the line of equilibria $\{\mathbf{T}_\alpha u^* \in \mathcal{Z}; \alpha \in \mathbb{R}\}$, and the manifold

$$\mathcal{M}_0 = \{\mathbf{T}_\alpha(u^* + v_0 + \mathbf{\Psi}(v_0)) \; ; \; v_0 \in \mathcal{E}_0', \; \alpha \in \mathbb{R}\} \subset \mathcal{Z}. \tag{4.30}$$

Then for differential equation (4.1) the following properties hold:

(i) The manifold $\mathcal{M}_0$ is locally invariant, i.e., if u is a solution of (4.1) satisfying $u(0) \in \mathcal{M}_0 \cap \mathcal{O}$ and $u(t) \in \mathcal{O}$ for all $t \in [0,T]$, then $u(t) \in \mathcal{M}_0$ for all $t \in [0,T]$.

(ii) $\mathcal{M}_0$ is locally attracting. Any solution of (4.1) staying in $\mathcal{O}$ for all $t \in \mathbb{R}$, tends exponentially towards $\mathcal{M}_0$.

We point out that in this situation the center manifold $\mathcal{M}_0$ attracts the solutions which stay close to the line of equilibria for all $t \in \mathbb{R}$. The asymptotic solutions are of the form

$$u = \mathbf{T}_\alpha(u^* + v_0 + \mathbf{\Psi}(v_0)),$$

with α and v_0 satisfying the reduced system

$$\frac{d\alpha}{dt} = g(v_0 + \mathbf{\Psi}(v_0)) \tag{4.31}$$

$$\frac{dv_0}{dt} = \mathbf{\Pi}_{\mathcal{H}}\mathbf{A}v_0 + \mathbf{P}_0'\left(\mathbf{\Pi}_{\mathcal{H}}\tilde{\mathbf{R}}(v_0 + \mathbf{\Psi}(v_0))\right) - \mathbf{P}_0'\left(g(v_0 + \mathbf{\Psi}(v_0))\mathbf{\Pi}_{\mathcal{H}}\boldsymbol{\tau}(v_0 + \mathbf{\Psi}(v_0))\right), \tag{4.32}$$

in which g is defined in (4.27) and $\mathbf{P}_0'$ is the spectral projector for the linear operator $\mathbf{L}' = \mathbf{\Pi}_{\mathcal{H}}\mathbf{A}$ defined as in Section 4.2.1. Furthermore, for such a solution we have that v_0 is a small bounded solution of the equation (4.32), whereas α given by (4.31) has bounded derivative and may grow linearly in t.

Similar results hold for the parameter-dependent equation (4.10) and for the nonautonomous equation (4.18).

4.4 Examples and Exercises

We end this chapter with some examples in which we apply the different variants of center manifold Theorem 4.8 presented in Section 4.3. In each example we show how to check the hypotheses and discuss the reduced system. In addition, these examples are such that $u = 0$ is a solution of the system for all values of the parameter(s), except for the example in Section 4.4.2, case V. This property allows us to use the result in Exercise 4.20, and so simplify some computations.

4.4.1 Burgers Model

We consider the initial boundary value problem

$$\frac{\partial \phi}{\partial t} = \frac{1}{\mathcal{R}}\frac{\partial^2 \phi}{\partial x^2} + \phi - \frac{\partial(\phi^2)}{\partial x} + U\phi, \tag{4.33}$$

$$\frac{dU}{dt} = -\frac{1}{\mathcal{R}}U - \int_0^1 \phi^2(x,t)dx, \tag{4.34}$$

$$\phi(0,t) = \phi(1,t) = 0, \tag{4.35}$$

where $\phi(x,t) \in \mathbb{R}$ and $U(t) \in \mathbb{R}$ for $(x,t) \in (0,1) \times \mathbb{R}$. This model equation, introduced by J. M. Burgers [5], is a one-dimensional model used for understanding instabilities in viscous fluid flows. In this system ϕ represents a velocity fluctuation, U is the induced perturbation on the mean basic flow, and $\mathcal{R}$ is the Reynolds number, proportional to the inverse of viscosity. The product $U\phi$ represents the interaction between the mean flow and the perturbation, the derivative of ϕ^2 represents inertial terms, and the integral represents Reynolds stresses.

Formulation as a First Order Equation We start by writing the problem (4.33)–(4.35) in the form (4.1), but now with linear part $\mathbf{L}$ depending upon the parameter $\mathcal{R}$, $\mathbf{L} = \mathbf{L}_\mathcal{R}$. We set

$$u = \begin{pmatrix} \phi \\ U \end{pmatrix}, \quad \mathbf{L}_\mathcal{R} u = \begin{pmatrix} \frac{1}{\mathcal{R}} \frac{\partial^2 \phi}{\partial x^2} + \phi \\ -\frac{1}{\mathcal{R}} U \end{pmatrix}, \quad \mathbf{R}(u) = \begin{pmatrix} -\frac{\partial(\phi^2)}{\partial x} + U\phi \\ -\int_0^1 \phi^2(x, \cdot)dx \end{pmatrix},$$

and choose the Hilbert space

$$\mathcal{X} = L^2(0,1) \times \mathbb{R}.$$

We include the boundary conditions (4.35) in the domain of definition $\mathcal{Z}$ of the operator $\mathbf{L}_\mathcal{R}$, by taking

$$\mathcal{Z} = (H^2(0,1) \cap H_0^1(0,1)) \times \mathbb{R}.$$

so that $\mathbf{R}(u) \in \mathcal{X}$ for $u \in \mathcal{Z}$. Notice that the system commutes with the symmetry $\mathbf{T}$ defined by

$$\mathbf{T} \begin{pmatrix} \phi(x) \\ U \end{pmatrix} = \begin{pmatrix} -\phi(1-x) \\ U \end{pmatrix},$$

which is an isometry in both $\mathcal{X}$ and $\mathcal{Z}$.

This formulation of the problem does not quite enter into the setting of center manifold theorems presented in the previous sections, because the linear operator depends upon the parameter $\mathcal{R}$. The next step consists in determining the spectrum of this operator in order to detect the "critical" values of the parameter $\mathcal{R}$, where its spectrum contains purely imaginary eigenvalues. These values are bifurcation points. Then we choose such a bifurcation point and apply the result in the parameter-dependent version of the center manifold theorem, Theorem 4.18, by taking $\mathbf{L}$ to be the operator $\mathbf{L}_\mathcal{R}$ at this bifurcation point.

Spectrum of the Linear Operator The linear operator $\mathbf{L}_\mathcal{R}$ is a closed operator in $\mathcal{X}$ with domain $\mathcal{Z}$. Since the domain $\mathcal{Z}$ is compactly embedded in $\mathcal{X}$, the operator $\mathbf{L}_\mathcal{R}$ has compact resolvent. Consequently, its spectrum consists of isolated eigenvalues, only, which all have finite algebraic multiplicity. In order to determine the spectrum we then solve the eigenvalue problem

$$\mathbf{L}_\mathcal{R} u = \lambda u, \quad u \in \mathcal{Z},$$

which is equivalent to the system

$$\phi'' + \mathcal{R}(1-\lambda)\phi = 0 \quad \phi(0) = \phi(1) = 0,$$
$$\left(\lambda + \frac{1}{\mathcal{R}}\right) U = 0.$$

The two equations in this system are decoupled, so that we can determine ϕ and U separately. The second equation gives the eigenvalue $\lambda_0 = -1/\mathcal{R}$, with eigenvector $(0,1)$, whereas by solving the first equation we find the sequence of eigenvalues $\lambda_k = 1 - k^2\pi^2/\mathcal{R}$, with eigenvectors $(\sin(k\pi x), 0)$ for $k \in \mathbb{N}^*$. Upon varying the parameter $\mathcal{R}$, we find that there is a sequence $(\mathcal{R}_k)_{k\in\mathbb{N}^*}$ of critical values of $\mathcal{R}$, where the part σ_0 of the spectrum of $\mathbf{L}_\mathcal{R}$ is not empty:

$$\mathcal{R}_k = k^2\pi^2, \quad k \in \mathbb{N}^*.$$

At each such value, $\sigma_0 = \{0\}$ and it is easy to check that the operators $\mathbf{L}_{\mathcal{R}_k}$ satisfy spectral Hypothesis 4.4. Furthermore, in each case the kernel of the operator $\mathbf{L}_{\mathcal{R}_k}$ is one-dimensional, spanned by the vector $(\sin(k\pi x), 0)$, so that 0 has geometric multiplicity one, and by setting the equation satisfied by a generalized eigenvector, it is not difficult to conclude that there is no solution and that the algebraic multiplicity is also one.

Checking Hypotheses 4.16 and 4.6 We restrict our analysis to the first bifurcation point $\mathcal{R} = \mathcal{R}_1 = \pi^2$. We set $\mu = \mathcal{R} - \mathcal{R}_1$ and write the system in the form (4.10) by taking

$$\mathbf{L} = \mathbf{L}_{\mathcal{R}_1}, \quad \mathbf{R}(u,\mu) = \mathbf{R}(u) + (\mathbf{L}_{\mathcal{R}_1+\mu} - \mathbf{L}_{\mathcal{R}_1})u.$$

Then $\mathbf{L}$ satisfies Hypothesis 4.16, and we have $\mathbf{R}(u,\mu) \in \mathcal{X}$ for $u \in \mathcal{Z}$, because of the term $(\mathbf{L}_{\mathcal{R}_1+\mu} - \mathbf{L}_{\mathcal{R}_1})u$, which belongs to $\mathcal{X}$. Since $\mathbf{R}(u)$ is quadratic, and

$$\|\mathbf{R}(u)\|_\mathcal{X} \leq C\|u\|_\mathcal{Z}^2 \text{ for all } u \in \mathcal{Z},$$

for some positive constant C, we have that $\mathbf{R} \in C^k(\mathcal{Z} \times \mathcal{V}_\mu, \mathcal{X})$ for any positive integer k, where $\mathcal{V}_\mu = \mathbb{R} \setminus \{\mathcal{R}_1\}$. Consequently, $\mathbf{R}$ satisfies Hypothesis 4.16.

It remains to check that Hypothesis 4.6 holds. For $f = (\psi, V) \in \mathcal{X}$, we have to show that the solution $u = (\phi, U) \in \mathcal{Z}$ of the system

$$(i\omega - 1)\phi - \frac{1}{\pi^2}\phi'' = \psi$$
$$\left(i\omega + \frac{1}{\pi^2}\right) U = V,$$

satisfies

$$\|u\|_\mathcal{X} = \left(\|\phi\|_{L^2(0,1)}^2 + |U|^2\right)^{1/2} \leq \frac{c}{|\omega|}\|f\|_\mathcal{X} = \frac{c}{|\omega|}\left(\|\psi\|_{L^2(0,1)}^2 + |V|^2\right)^{1/2},$$

for $|\omega| \geq \omega_0$ and some positive constant c. First, from the second equation we immediately find

$$|U| = \frac{\pi^2}{\sqrt{1 + \pi^4\omega^2}}|V|, \tag{4.36}$$

whereas for the solution ϕ of the first equation, we can make use of the fact that we know that this solution exists and belongs to $H^2(0,1) \cap H_0^1(0,1)$ for $\psi \in L^2(0,1)$, when $\omega \neq 0$, since any $i\omega \neq 0$ belongs to the resolvent set of $\mathbf{L}_{\mathcal{R}_1}$. Then multiplying the equation by $\bar{\phi}$, integrating over $(0,1)$, and integrating once by parts we obtain

$$(i\omega - 1)\|\phi\|_{L^2(0,1)}^2 + \frac{1}{\pi^2}\|\phi'\|_{L^2(0,1)}^2 = \int_0^1 \psi(x)\bar{\phi}(x)\,dx.$$

Upon taking the imaginary parts of both sides of this equality we find

$$\omega\|\phi\|^2_{L^2(0,1)} = Im\int_0^1 \psi(x)\bar{\phi}(x)\,dx,$$

so that

$$|\omega|\|\phi\|^2_{L^2(0,1)} \leq \int_0^1 |\psi(x)\bar{\phi}(x)|\,dx \leq \|\psi\|_{L^2(0,1)}\|\phi\|_{L^2(0,1)}.$$

Consequently,

$$\|\phi\|_{L^2(0,1)} \leq \frac{1}{|\omega|}\|\psi\|_{L^2(0,1)},$$

which together with (4.36) gives the desired estimate and proves that Hypothesis 4.6 holds.

Center Manifold Hypotheses 4.16, 4.4, and 4.6 being satisfied, we can now apply center manifold Theorem 4.18. Since 0 is a simple eigenvalue, the space $\mathcal{E}_0$ is one-dimensional, which gives us the family of one-dimensional center manifolds $\mathcal{M}_0(\mu)$, as in (4.23), for sufficiently small μ. We have that $\mathbf{L}_0 u_0 = 0$, so that the linear term in the reduced system (4.6) vanishes. Further denote by ξ_0 the eigenvector

$$\xi_0 = (\sin(\pi x), 0)$$

which spans $\mathcal{E}_0$, and write

$$u_0(t) = A(t)\xi_0 \in \mathcal{E}_0, \quad A(t) \in \mathbb{R}.$$

Replacing this formula in the reduced system (4.15) we obtain a first order ODE for A,

$$\frac{dA}{dt} = f_0(A, \mu),$$

with $f_0(A, \mu) = O(|A|(|\mu| + |A|))$, as $(A, \mu) \to (0, 0)$.

Now, recall that the system commutes with the symmetry $\mathbf{T}$, so that the result in Theorem 4.29 holds, as well. Then the vector field in the reduced system commutes with the induced symmetry $\mathbf{T}_0$ on $\mathcal{E}_0$. Since $\mathbf{T}\xi_0 = -\xi_0$, this symmetry acts on A through $A \mapsto -A$. In particular, this shows that the vector field f_0 is odd in A, so that we may write

$$\frac{dA}{dt} = a\mu A + bA^3 + O(|A|(|\mu|^2 + A^4)).$$

We expect to find here a pitchfork bifurcation (see Section 2.2). In order to analyze this bifurcation we compute the coefficients a and b.

Pitchfork Bifurcation The coefficient a can be computed with the help of the result in Exercise 4.20, which shows that $\partial f_0/\partial A(0, \mu)$ is the eigenvalue of $\mathbf{L}_{\mathcal{R}_1+\mu}$ vanishing at $\mu = 0$. This latter eigenvalue is

$$\lambda_1 = 1 - \frac{\pi^2}{\mathcal{R}_1 + \mu} = \frac{\mu}{\pi^2} - \frac{\mu^2}{\pi^4} + O(|\mu|^3),$$

so that we find

$$a = \frac{1}{\pi^2}.$$

Next, in order to compute b we write for u on the center manifold

$$u(t) = A(t)\xi_0 + \mathbf{\Psi}(A(t), \mu), \tag{4.37}$$

in which $u_0(t) = A(t)\xi_0$ and $\mathbf{\Psi}$ is the reduction function. Recall that $\mathbf{R}(u, 0) = \mathbf{R}(u)$ is quadratic, so that we may write

$$\mathbf{R}(u,0) = \mathbf{R}_2(u,u), \quad \mathbf{R}_2(u,v) = \begin{pmatrix} -\frac{\partial(\phi\psi)}{\partial x} + \frac{1}{2}U\psi + \frac{1}{2}V\phi \\ -\int_0^1 \phi(x,\cdot)\psi(x,\cdot)dx \end{pmatrix},$$

where $v = (\psi, V)$. We set $\mu = 0$ in the following calculations, and consider the expansion

$$\mathbf{\Psi}(A,0) = A^2\mathbf{\Psi}_2 + A^3\mathbf{\Psi}_3 + O(A^4),$$

in which $\mathbf{T\Psi}_2 = \mathbf{\Psi}_2$, and $\mathbf{T\Psi}_3 = -\mathbf{\Psi}_3$, because $\mathbf{\Psi}$ commutes with the symmetry $\mathbf{T}$. Now we substitute u from (4.37) into

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}_2(u,u), \tag{4.38}$$

and taking into account that

$$\frac{dA}{dt} = bA^3 + O(|A|^5)$$

when $\mu = 0$, we identify the powers of A in this equality. At orders $O(A^2)$ and $O(A^3)$, we find, respectively,

$$\begin{aligned} \mathbf{L\Psi}_2 &= -\mathbf{R}_2(\xi_0, \xi_0), \\ \mathbf{L\Psi}_3 &= -2\mathbf{R}_2(\xi_0, \mathbf{\Psi}_2) + b\xi_0. \end{aligned}$$

A necessary condition for solving these equations is that the right hand sides of both equalities lie in the range of $\mathbf{L}$, or equivalently, lie in the space orthogonal to the kernel of the adjoint of $\mathbf{L}$. A direct calculation shows that here $\mathbf{L}^* = \mathbf{L}$, i.e., $\mathbf{L}$ is self-adjoint, so that its kernel is spanned by ξ_0. Further, recall that $\mathbf{\Psi}(A, \mu)$ belongs to $\mathcal{Z}_h$, the space defined by $\mathcal{Z}_h = (\mathbb{I} - \mathbf{P}_0)\mathcal{Z}$, where $\mathbf{P}_0$ is the spectral projection onto $\mathcal{E}_0$, associated with σ_0. It is this property which allows one to uniquely determine $\mathbf{\Psi}_2$ and $\mathbf{\Psi}_3$ from the equalities above. However, in this particular example we can get the desired result without explicitly computing the projection $\mathbf{P}_0$.

First,

$$\mathbf{R}_2(\xi_0, \xi_0) = \begin{pmatrix} -\pi \sin(2\pi x) \\ -\frac{1}{2} \end{pmatrix},$$

which is clearly orthogonal to ξ_0 in $\mathcal{X}$, and a direct calculation gives

$$\mathbf{\Psi}_2 = \begin{pmatrix} -\frac{\pi}{3}\sin(2\pi x) \\ -\frac{\pi^2}{2} \end{pmatrix} + \alpha\xi_0$$

for some $\alpha \in \mathbb{R}$. Now, recall that $\mathbf{T}\mathbf{\Psi}_2 = \mathbf{\Psi}_2$, which together with the fact that $\mathbf{T}\xi_0 = -\xi_0$, implies that $\alpha = 0$. Next, we compute

$$2\mathbf{R}_2(\xi_0, \mathbf{\Psi}_2) = \begin{pmatrix} \pi^2 \sin(3\pi x) - \frac{5\pi^2}{6} \sin(\pi x) \\ 0 \end{pmatrix}.$$

The solvability condition for the second equation is

$$0 = \langle b\xi_0 - 2\mathbf{R}_2(\xi_0, \mathbf{\Psi}_2), \xi_0 \rangle = \frac{1}{2}b + \frac{5\pi^2}{12},$$

so that

$$b = -\frac{5\pi^2}{6}.$$

Summarizing, the reduced equation is

$$\frac{dA}{dt} = \frac{1}{\pi^2}\mu A - \frac{5\pi^2}{6}A^3 + O(|A|(|\mu|^2 + |A|^4)),$$

in which the right hand side is odd in A. According to the result in Theorem 2.9, we have here a *supercritical pitchfork bifurcation*, in which a pair of steady solutions emerges from 0 as $\mathcal{R}$ crosses $\mathcal{R}_1$. These steady solutions are stable, whereas the trivial solution $A = 0$ is stable for $\mathcal{R} < \mathcal{R}_1$ and unstable for $\mathcal{R} > \mathcal{R}_1$ (see Figure 4.1).

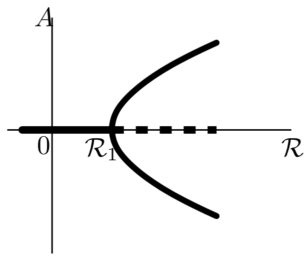


Figure 4.1: Supercritical pitchfork bifurcation, which occurs at the first bifurcation point $\mathcal{R}_1 = \pi^2$ in the Burgers model.

Exercise 4.35. *Consider the integro-differential equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1 - e^{-\nu u} - K \int_0^\pi u(x,t)dx,$$
$$\frac{\partial u}{\partial x}\Big|_{x=0} = \frac{\partial u}{\partial x}\Big|_{x=\pi} = 0,$$

where $u(x,t) \in \mathbb{R}$ for $(x,t) \in (0,\pi) \times \mathbb{R}$, and K, ν are real parameters.

(i) *Check that $u = 0$ is a solution of this problem for all K and ν. Write the system in the form (4.1) with linear operator $\mathbf{L} = \mathbf{L}_{K,\nu}$, depending upon the two parameters K and ν.*

(ii) *Show that the system is equivariant under the symmetry* $\mathbf{T}$ *defined by*

$$\mathbf{T}u(x,t) = u(\pi - x, t).$$

(iii) *Show that the spectrum of* $\mathbf{L}_{K,\nu}$ *is a discrete set,* $\sigma = \{\lambda_n \in \mathbb{R}; n \in \mathbb{N}\}$*, consisting of the eigenvalues*

$$\lambda_0 = \nu - K\pi, \quad \lambda_n = \nu - n^2, \quad n = \mathbb{N}^*,$$

with associated eigenvectors

$$\xi_n = \cos(nx), \quad n \in \mathbb{N}.$$

Give the action of the symmetry $\mathbf{T}$ *on these eigenvectors.*

(iv) *Assume* $K\pi > 1$*, and set* $\nu = 1 + \mu$*. Write the system in the form (4.10) and show that it possesses a center manifold of dimension 1. Show that the reduced equation takes the form*

$$\frac{dA}{dt} = \mu A + bA^3 + O(|A|(|\mu|^2 + |A|^4)), \quad b = \frac{1}{6} + \frac{1}{4(K\pi - 1)} > 0.$$

(Notice that the coefficient b *tends towards* ∞ *when* $K\pi \to 1$*. This is due to the invalidity of the study when* $K\pi$ *is close to 1, since at* $K\pi = 1$ *there are two "critical" eigenvalues,* λ_0 *and* λ_1*, instead of only one for* $K\pi > 1$*.)*

(v) *Consider* $K\pi$ *and* ν *close to 1, and set* $\mu = \nu - 1$ *and* $\varepsilon = \nu - K\pi$*. Write the system in the form (4.10) and show that it possesses a center manifold of dimension 2. Show that the reduced system is given by*

$$\begin{aligned} \frac{dA}{dt} &= \mu A - AB + \frac{1}{6}A^3 + h.o.t. \\ \frac{dB}{dt} &= (\mu - \varepsilon)B - \frac{1}{4}A^2 - \frac{1}{2}B^2 + h.o.t., \end{aligned}$$

in which the first component of the vector field is odd in A*, and the second component is even in* A*. Here and in the remainder of this book "h.o.t." denotes higher order terms.*

4.4.2 Swift–Hohenberg Equation

We consider the Swift–Hohenberg equation (SHE)

$$\frac{\partial u}{\partial t} = -\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u + \mu u - u^3, \tag{4.39}$$

where $u = u(x,t) \in \mathbb{R}$ for $(x,t) \in \mathbb{R}^2$, and μ is a real parameter. The Swift–Hohenberg equation arises as a model for hydrodynamical instabilities. We refer to [10] for a detailed analysis of this equation.

Notice that $u = 0$ is a solution of (4.39) and that the equation is invariant under spatial translations $x \mapsto x + \alpha$, $\alpha \in \mathbb{R}$, and the reflections $x \mapsto -x$ and $u \mapsto -u$.

Linear Stability Analysis We first analyze the linear stability of the trivial solution $u = 0$. We look for solutions of the form

$$u(x,t) = \widehat{u}e^{ikx+\lambda t}, \tag{4.40}$$

where k is a real wavenumber and λ and $\widehat{u}$ may be complex numbers, of the linearized SHE

$$\frac{\partial u}{\partial t} = -\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u + \mu u.$$

Inserting (4.40) into the linearized equation gives the linear dispersion relation

$$\lambda(\mu,k) = \mu - (1-k^2)^2. \tag{4.41}$$

The solution $u = 0$ is linearly stable (resp., unstable) with respect to the mode e^{ikx} if $Re\lambda(\mu,k) < 0$ (resp., $Re\lambda(\mu,k) > 0$).

The dispersion relation (4.41) shows that $\lambda(\mu,k)$ is real for all k and μ. For a fixed μ, the solution $u = 0$ is stable with respect to all modes e^{ikx} for which $\mu < (1-k^2)^2$, and unstable with respect to all modes for which $\mu > (1-k^2)^2$. The modes e^{ikx} such that $(1-k^2)^2 = \mu$ are the critical modes at the threshold from stability to instability. We plot in Figure 4.2 the curve $\lambda(\mu,k) = 0$. This shows that, upon increasing μ, the first critical

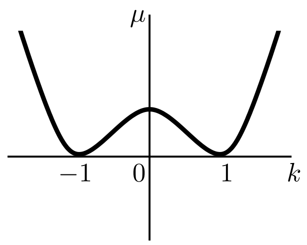


Figure 4.2: Critical curve $\lambda(\mu,k) = 0$ for the Swift–Hohenberg equation.

modes, $k = \pm 1$, occur at $\mu = 0$. These modes correspond to 2π-periodic solutions $e^{\pm ix}$ of the linearized equation, at the threshold of linear instability. We therefore expect spatially 2π-periodic solutions to play a particular role in the dynamics of the equation, and restrict ourselves to this type of solutions in our analysis.

Center Manifolds We write the equation in the form (4.1), with linear operator $\mathbf{L} = \mathbf{L}_\mu$ depending upon the parameter μ, by setting

$$\mathbf{L}_\mu = -\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 + \mu, \quad \mathbf{R}(u) = -u^3,$$

and choosing the spaces of 2π-periodic functions

$$\mathcal{X} = L^2_{\text{per}}(0,2\pi), \mathcal{Z} = H^4_{\text{per}}(0,2\pi).$$

Then $\mathbf{L}_\mu$ is a closed operator in $\mathcal{X}$ with domain $\mathcal{Z}$, and $\mathbf{R}$ is a cubic map in $\mathcal{Z}$, satisfying

$$\|\mathbf{R}(u)\|_{\mathcal{Z}} \leq C\|u\|_{\mathcal{Z}}^3,$$

so that $\mathbf{R} \in C^k(\mathcal{Z})$ for any positive integer k.

Next, we compute the spectrum of $\mathbf{L}_\mu$. As for the operator in the previous example, Section 4.4.1, the domain $\mathcal{Z}$ of $\mathbf{L}_\mu$ is compactly embedded in $\mathcal{X}$, so that $\mathbf{L}_\mu$ has a compact resolvent. Consequently, its spectrum consists only of isolated eigenvalues with finite multiplicities. Since we work in spaces of 2π-periodic functions, we can use Fourier analysis to solve the eigenvalue problem and conclude that

$$\sigma = \{\lambda_n = \mu - (1-n^2)^2 \ ; \ n \in \mathbb{N}\}.$$

All these eigenvalues are real, and there is a sequence $(\mu_n = (1-n^2)^2)_{n\in\mathbb{N}}$ of values of μ for which 0 is an eigenvalue of $\mathbf{L}_\mu$. The smallest value, $\mu_1 = 0$, is the one at which the solution $u = 0$ loses its stability when increasing μ. We apply center manifold Theorem 4.18 for values of μ close to this critical value $\mu_1 = 0$.

We proceed as in the example in Section 4.4.1 and first rewrite the equation in the form (4.10), with

$$\mathbf{L} = \mathbf{L}_0, \quad \mathbf{R}(u,\mu) = \mathbf{R}(u) + (\mathbf{L}_\mu - \mathbf{L}_0)u.$$

From the arguments above it follows that $\mathbf{L}$ and $\mathbf{R}$ satisfy Hypothesis 4.16 and that Hypothesis 4.4 holds with $\sigma_0 = \{0\}$. Furthermore, 0 is an eigenvalue with geometric multiplicity two, with associated eigenvectors $e^{\pm ix}$, and by arguing as in previous example, one can show that its algebraic multiplicity is two as well. (Alternatively, notice that $\mathbf{L}_\mu$ is self-adjoint in $\mathcal{X}$ so that its eigenvalues are all semisimple. In particular, 0 is then a double eigenvalue of $\mathbf{L}$.) Finally, Hypothesis 4.6 can be checked as in the example in Section 4.4.1. Applying Theorem 4.18, we conclude that the equation possesses a two-dimensional center manifold for μ sufficiently small.

Symmetries An important role in this example is played by the different symmetries of the SHE mentioned above. The invariance under spatial translations $x \mapsto x+\alpha$, $\alpha \in \mathbb{R}$, and the reflections $x \mapsto -x$ and $u \mapsto -u$ imply that the equation is equivariant with respect to the isometries defined by

$$(\mathbf{T}_\alpha u)(x) = u(x+\alpha), \quad \alpha \in \mathbb{R}, \quad (\mathbf{T}u)(x) = u(-x), \quad (\mathbf{U}u)(x) = -u(x).$$

All these symmetries, $(\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}}$, $\mathbf{T}$, and $\mathbf{U}$, satisfy Hypothesis 4.28. Consequently, the result in Theorem 4.29 holds with any of these symmetries. The family $(\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}}$ also satisfies Hypothesis 4.32. However, we haven't in this case a nontrivial equilibrium satisfying Hypothesis 4.33, so that we cannot argue as for Theorem 4.34 in this example.

In addition, notice that

$$\mathbf{T}_\alpha = \mathbf{T}_{\alpha+2\pi}, \quad \mathbf{T}\mathbf{T}_\alpha = \mathbf{T}_{-\alpha}\mathbf{T}, \quad \mathbf{U}\mathbf{T}_\alpha = \mathbf{T}_\alpha\mathbf{U}, \quad \alpha \in \mathbb{R}.$$

The first equality is a consequence of the fact that we restrict our analysis to 2π-periodic functions in x. In particular, the first two equalities show that (4.39) is equivariant under the representation of the group $O(2)$ by $(\mathbf{T}, (\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}/2\pi\mathbb{Z}})$.

Steady $O(2)$ Bifurcation We discuss now the reduced system given by Theorems 4.18 and 4.29. Recall that the subspace $\mathcal{E}_0$ is two-dimensional, spanned by the complex conjugated eigenvector $\zeta = e^{ix}$ and $\overline{\zeta} = e^{-ix}$, so that it is convenient in this case to write

$$u_0 = A\zeta + \bar{A}\,\overline{\zeta}, \quad A(t) \in \mathbb{C},$$

for real-valued $u_0(t) \in \mathcal{E}_0$. Then we set for the real-valued solutions on the center manifold

$$u = A\zeta + \overline{A\zeta} + \mathbf{\Psi}(A, \bar{A}, \mu), \quad A(t) \in \mathbb{C},$$

where $\mathbf{\Psi}(A(t), \bar{A}(t), \mu) \in \mathcal{Z}_h$. The reduced equation reads

$$\frac{dA}{dt} = f(A, \bar{A}, \mu), \tag{4.42}$$

together with the complex conjugated equation for $\bar{A}$. In addition, since the original equation is equivariant under the actions of $\mathbf{T}_\alpha$ and $\mathbf{T}$, by the result in Theorem 4.29, we have that the reduced vector field $(f, \overline{f})$ is equivariant under the actions of the induced symmetries. Since

$$\mathbf{T}_\alpha\zeta = e^{i\alpha}\zeta, \quad \mathbf{T}_\alpha\overline{\zeta} = e^{-i\alpha}\overline{\zeta}, \quad \mathbf{T}\zeta = \overline{\zeta}, \quad \mathbf{T}\overline{\zeta} = \zeta,$$

the action of the induced symmetries on the pair $(A, \bar{A})$ is given by the 2×2-matrices

$$\mathbf{T}_\alpha \ : \ \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \quad \mathbf{T} \ : \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This shows that we are in the setting of the study made in Section 3.4, on steady bifurcations with $O(2)$ symmetry. Consequently, we have that

$$f(A, \bar{A}, \mu) = Ag(|A|^2, \mu),$$

where the function g is of class C^{k-1} in $(A, \bar{A}, \mu)$ and real-valued. We consider the Taylor expansion of g and write

$$\frac{dA}{dt} = aA\mu + bA|A|^2 + O(|A|(|\mu|^2 + |A|^4)).$$

In polar coordinates, for $A = re^{i\phi}$, this gives the system (3.42)–(3.43) studied in Section 3.4.

We now compute the coefficients a and b in order to determine the nature of this bifurcation. For this we proceed as in the previous example in Section 4.4.1. First, using the result in the Exercise 4.20, we obtain

$$\frac{\partial f}{\partial A}(0, \mu) = \lambda_1 = \mu,$$

so that

$$a = 1.$$

Next, we set $\mu = 0$ in the following calculations and consider the expansion of the reduction function $\mathbf{\Psi}$,

$$\mathbf{\Psi}(A, \bar{A}, 0) = \sum_{p,q} \mathbf{\Psi}_{pq} A^p \overline{A}^q.$$

Here $\mathbf{\Psi}_{qp} \in \mathcal{Z}_h$ are such that

$$\mathbf{\Psi}_{qp} = \overline{\mathbf{\Psi}_{pq}}, \quad \mathbf{\Psi}_{00} = \mathbf{\Psi}_{10} = \mathbf{\Psi}_{01} = 0.$$

The first equality shows that $\mathbf{\Psi}$ is real-valued, whereas the last equalities come from (4.22). Furthermore, from the equivariance of the equation with respect to $\mathbf{U}$, we conclude that $\mathbf{\Psi}(-A, -\overline{A}, 0) = -\mathbf{\Psi}(A, \overline{A}, 0)$ for all A, and thus $\mathbf{\Psi}_{pq} = 0$ when $p+q$ is even. Summarizing, we find the expansion

$$\mathbf{\Psi}(A, \bar{A}, 0) = \mathbf{\Psi}_{30}A^3 + \mathbf{\Psi}_{03}\bar{A}^3 + \mathbf{\Psi}_{21}A^2\bar{A} + \mathbf{\Psi}_{12}A\bar{A}^2 + O(|A|^5),$$

where $\mathbf{\Psi}_{03} = \overline{\mathbf{\Psi}_{30}}$ and $\mathbf{\Psi}_{12} = \overline{\mathbf{\Psi}_{21}}$.

Now by arguing as in the calculation of the coefficient b in the example in Section 4.4.1, we obtain the equalities

$$\begin{aligned} \mathbf{L}\mathbf{\Psi}_{30} &= e^{3ix}, \\ \mathbf{L}\mathbf{\Psi}_{21} &= 3e^{ix} + be^{ix}. \end{aligned}$$

The solvability condition for the second equation gives

$$b = -3.$$

Summarizing, the reduced equation is

$$\frac{dA}{dt} = \mu A - 3A|A|^2 + O(|A|(|\mu|^2 + |A|^4)), \tag{4.43}$$

and the reduced vector field possesses an $O(2)$ equivariance, just as in Hypothesis 3.13. According to the result in Theorem 3.17, we have here a *steady bifurcation with $O(2)$ symmetry*, in which a family $(A_\alpha)_{\alpha\in\mathbb{R}/2\pi\mathbb{Z}}$ of stable equilibria emerges from 0, as μ crosses 0. A direct calculation gives

$$A_\alpha = \sqrt{\frac{\mu}{3}}e^{i\alpha} + O(|\mu|^{3/2})$$

for $\mu > 0$, and the corresponding family of steady 2π-periodic solutions of SHE,

$$u_\alpha(x) = 2\sqrt{\frac{\mu}{3}}\cos(x+\alpha) + O(|\mu|^{3/2}). \tag{4.44}$$

We point out that $u_\alpha = \mathbf{T}_\alpha u_0$, so that the solutions in this family are obtained by spatially translating u_0.

Remark 4.36. *These steady 2π-periodic solutions of the SHE are called* roll solutions. *Actually, such solutions exist for a range of periods close to 2π, for any sufficiently small μ. One can prove the existence of all these rolls in a similar way. Looking for periodic solutions of the SHE with wavenumbers k close to 1, instead of wavenumbers $k = 1$, only, and normalizing the period to 2π in the equation, one finds an equation having an additional parameter, the wavenumber k. The normalization of the period allows us to use the same function spaces $\mathcal{X}$ and $\mathcal{Z}$, and this reduction procedure can be performed with two parameters, k close to 1 and μ small.*

Symmetry Breaking We briefly discuss here several scenarios in which we perturb the Swift–Hohenberg equation, by adding a small term, in such a way that one, or more, of the symmetries of the SHE is broken. We are interested in the effect of the perturbation on the reduced equation (4.43).

I. First we consider the perturbed equation obtained by adding the term εu^2 in the right hand side of the SHE, with ε a small real parameter. This term breaks the equivariance of the equation with respect to the symmetry $\mathbf{U}$ but preserves the $O(2)$ equivariance with respect to $(\mathbf{T}, (\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}/2\pi\mathbb{Z}})$. The center manifold analysis remains the same, up to the equivariance in $\mathbf{U}$, which is lost, and to the appearance of the additional small parameter ε. However, this parameter does not play a role in checking the different hypotheses, its effect being that now the reduced vector field $(f, \bar{f})$ depends upon ε as well. Since the $O(2)$ equivariance is preserved, we still have the particular form

$$f(A, \bar{A}, \mu, \varepsilon) = Ag(|A|^2, \mu, \varepsilon),$$

with g of class C^{k-1} and real-valued.

Notice that at $\varepsilon = 0$ we find exactly the reduced vector field obtained for the unperturbed equation. Furthermore, we have here a new symmetry, which is the invariance of the SHE under $(u, \varepsilon) \mapsto (-u, -\varepsilon)$. It is then straightforward to check that this induces the invariance of the reduced equation under the action of $(A, \varepsilon) \mapsto (-A, -\varepsilon)$. In particular, this shows that the map g above is even in ε. This fact is useful in the computation of the Taylor expansion of g.

II. Next, we add the term $\varepsilon\partial u/\partial x$ in the right hand side of the SHE, with ε a small real parameter. This situation actually reduces to the unperturbed SHE, by the change of variables $u(x, t) = \widetilde{u}(x + \varepsilon t, t)$. It is easy to see that u is a solution of the perturbed SHE if and only if $\widetilde{u}$ is a solution of the unperturbed SHE. In particular, our previous analysis gives us in this case the family of traveling wave solutions $u_\alpha(x + \varepsilon t)$, with u_α the steady 2π-periodic solution in (4.44). These traveling waves have small speeds $-\varepsilon$, are 2π-periodic in the spatial variable x, and are periodic in time with large period $2\pi/\varepsilon$.

Our interest in considering this example is to see the effect of such a term on the different symmetries of the SHE and then on the reduced system. This term breaks the symmetry $\mathbf{T}$, but preserves the symmetries $\mathbf{T}_\alpha$ and $\mathbf{U}$. In particular, instead of an $O(2)$ equivariance we have now an $SO(2)$ equivariance. However, one can argue as in Section 3.4 and conclude that the map f in the reduced system is of the form

$$f(A, \bar{A}, \mu, \varepsilon) = Ag(|A|^2, \mu, \varepsilon),$$

with g of class C^{k-1}, and complex-valued but not necessarily real-valued anymore.

In this situation, we have the additional invariance of the SHE under $(x, \varepsilon) \mapsto (-x, -\varepsilon)$. On the center manifold, this induces the symmetry acting by $(A, \varepsilon) \mapsto (\bar{A}, -\varepsilon)$, so that g satisfies

$$g(|A|^2, \mu, \varepsilon) = \overline{g(|A|^2, \mu, -\varepsilon)}.$$

Consequently, the real part g_r of g is even in ε, whereas the imaginary part g_i of g is odd in ε. This leads to the equation

$$\frac{dA}{dt} = (\mu + c\varepsilon^2 + id\varepsilon)A - 3A|A|^2 + h.o.t.,$$

which in polar coordinates $A = re^{i\phi}$ reads

$$\begin{aligned}\frac{dr}{dt} &= (\mu + c\varepsilon^2)r - 3r^3 + h.o.t.\\ \frac{d\phi}{dt} &= d\varepsilon + h.o.t.\end{aligned} \tag{4.45}$$

Here the real coefficients c and d can be computed explicitly, just as the coefficients a and b in (4.42), and we have used the fact that the reduced system at $\varepsilon = 0$ is the same as the reduced system found for the unperturbed equation. It is then straightforward to find the solutions

$$r_0(\mu, \varepsilon^2) = \left(\frac{\mu + c\varepsilon^2}{3}\right)^{1/2} + h.o.t., \quad \phi_0 = \omega t + \alpha, \quad \omega = d\varepsilon + h.o.t.,$$

with any $\alpha \in \mathbb{R}$. These give the solutions of the perturbed SHE equation

$$u(x,t) = 2r_0(\mu, \varepsilon^2)\cos(x + \omega t + \alpha) + h.o.t..$$

The lowest order term in this solution is clearly a traveling wave, with speed $-\omega$. A careful use of the symmetries mentioned above, together with the invariance of the equation under translations in the time t, allows us to show that these solutions are indeed traveling waves.

Exercise 4.37. *Show that $c = 0$ and $d = 1$ in the reduced system (4.45).*

III. Consider now the additional term $\varepsilon u \partial u/\partial x$ on the right hand side of the SHE. This term breaks the symmetries $\mathbf{T}$ and $\mathbf{U}$, but preserves the composed symmetry $\widetilde{\mathbf{T}} = \mathbf{T} \circ \mathbf{U}$ and the family $(\mathbf{T}_\alpha)_{\alpha\in\mathbb{R}}$. Consequently, we still have an $O(2)$ equivariance of the system, but now with $\widetilde{\mathbf{T}}$ instead of $\mathbf{T}$. The action of $\widetilde{\mathbf{T}}$ on the pair $(A, \bar{A})$ is given by the 2×2-matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

However this does not change the form of the reduced equation, the map f being again of the form

$$f(A, \bar{A}, \mu, \varepsilon) = Ag(|A|^2, \mu, \varepsilon).$$

In addition, we have here the symmetry $(u, \varepsilon) \mapsto (-u, -\varepsilon)$, which implies that

$$g(|A|^2, \mu, -\varepsilon) = g(|A|^2, \mu, \varepsilon).$$

IV. We introduce now an additional term $\varepsilon_1 u\partial u/\partial x + \varepsilon_2 u^2$, in which we have two small parameters ε_1 and ε_2. This term breaks the symmetries $\mathbf{T}$, $\mathbf{U}$, and also $\widetilde{\mathbf{T}} = \mathbf{T} \circ \mathbf{U}$, but preserves the symmetries $\mathbf{T}_\alpha$, $\alpha \in \mathbb{R}$. Consequently, we still have an $SO(2)$ equivariance, just as in the case **II**, which allows us to conclude that the map f in the reduced system is of the form

$$f(A, \bar{A}, \mu, \varepsilon_1, \varepsilon_2) = Ag(|A|^2, \mu, \varepsilon_1, \varepsilon_2),$$

with g of class C^{k-1} and complex-valued.

In addition, we now find the new symmetries

$$(u, \varepsilon_1, \varepsilon_2) \mapsto (-u, -\varepsilon_1, -\varepsilon_2), \quad (u(x), \varepsilon_1, \varepsilon_2) \mapsto (u(-x), -\varepsilon_1, \varepsilon_2).$$

Their action on $(A, \bar{A})$ is given by

$$(A, \bar{A}, \varepsilon_1, \varepsilon_2) \mapsto (-A, -\overline{A}, -\varepsilon_1, -\varepsilon_2), \quad (A, \bar{A}, \varepsilon_1, \varepsilon_2) \mapsto (\bar{A}, A, -\varepsilon_1, \varepsilon_2).$$

We can then conclude that the map g satisfies

$$g(|A|^2, \mu, \varepsilon_1, \varepsilon_2) = g(|A|^2, \mu, -\varepsilon_1, -\varepsilon_2), \quad g(|A|^2, \mu, \varepsilon_1, \varepsilon_2) = \overline{g(|A|^2, \mu, -\varepsilon_1, \varepsilon_2)},$$

so that the reduced equation is

$$\frac{dA}{dt} = \mu A - 3A|A|^2 + (c_1\varepsilon_1^2 + id\varepsilon_1\varepsilon_2 + c_2\varepsilon_2^2)A|A|^2 + h.o.t..$$

In polar coordinates $A = re^{i\phi}$, we find the system

$$\begin{aligned} \frac{dr}{dt} &= \mu r - 3r^3 + (c_1\varepsilon_1^2 + c_2\varepsilon_2^2)r^3 + h.o.t. \\ \frac{d\phi}{dt} &= d\varepsilon_1\varepsilon_2 r^2 + h.o.t.. \end{aligned}$$

By arguing as for the system (4.45) in case **II**, one can show in this case the existence of bifurcating traveling waves with speeds of order $O(\mu\varepsilon_1\varepsilon_2)$.

Exercise 4.38. *Show that* $c_1 = -1/9$, $d = 4/3$, *and* $c_2 = 20/9$ *in the reduced system.*

V. Consider now the case of an inhomogeneous additional term $\varepsilon h(x)$, on the right hand side of the SHE, where $h : \mathbb{R} \to \mathbb{R}$ is an even 2π-periodic function and ε a small parameter, again. Notice that in this case the trivial solution $u = 0$ is no longer a solution for $\varepsilon \neq 0$.

This term now breaks the translation invariance $\mathbf{T}_\alpha$, $\alpha \in \mathbb{R}$, and the reflection $\mathbf{U}$, but preserves the symmetry $\mathbf{T}$. As in the previous cases we find a two-dimensional center manifold and a reduced equation of the form

$$\frac{dA}{dt} = f(A, \bar{A}, \mu, \varepsilon)$$

for $A(t) \in \mathbb{C}$. At $\varepsilon = 0$, the map f is the one obtained for the unperturbed equation,

$$f(A, \bar{A}, \mu, 0) = Ag(|A|^2, \mu) = \mu A - 3A|A|^2 + h.o.t.,$$

whereas for $\varepsilon \neq 0$ the equivariance with respect to $\mathbf{T}$ implies that

$$f(A, \bar{A}, \mu, \varepsilon) = \overline{f(\bar{A}, A, \mu, \varepsilon)}.$$

Consequently, the reduced equation is of the form

$$\frac{dA}{dt} = c\varepsilon + \mu A - 3A|A|^2 + h.o.t.,$$

where c is a real constant. Notice that the constant term on the right hand side of this equation is real, because of the property of f above, and nonzero, since $u = 0$ is no longer a solution of the perturbed equation.

Exercise 4.39. *Show that the coefficient* c *in the reduced system is given by*

$$c = \frac{1}{2\pi}\int_0^{2\pi} h(x)\cos x dx.$$

Remark 4.40 (Steady solutions)**.** *Notice that the steady solutions of this system are easy to compute. They are real,* $A = A_r$*, with* A_r *satisfying*

$$c\varepsilon + A_r(\mu - 3A_r^2) + h.o.t. = 0.$$

We plot in Figure 4.3 the bifurcation diagram for the steady solutions of this reduced equation. As for the stability of these steady solutions, it can be determined from the eigenvalues of the linearized vector field at $A = A_r$*. A direct calculation gives the two eigenvalues* $\mu - 9A_r^2 + h.o.t.$ *and* $\mu - 3A_r^2 + h.o.t.$*. In particular, in the case represented in the bifurcation diagram in Figure 4.3, the upper branch is stable (both eigenvalues are negative), while the lower branch is unstable (at least one eigenvalue is positive). We point out that this result differs from the classical result occuring in a perturbed pitchfork bifurcation. Notice that one eigenvalue is 0 at the turning point of the lower branch, but that this does not change the stability here, because of the second eigenvalue. Moreover, observe that all these steady solutions are symmetric, invariant under* $\mathbf{T}$*, since they are real.*

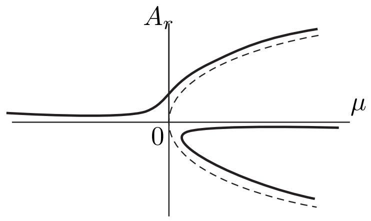


Figure 4.3: Bifurcation diagram in the (μ, A_r)-plane for the steady solutions of the reduced system in the SHE perturbed by an inhomogeneity $\varepsilon h(x)$ in the case $c\varepsilon > 0$. The solid lines represent the branches of steady solutions for a fixed, small ε, whereas the dashed lines represent the branch of steady solutions for $\varepsilon = 0$.

VI. Finally, we consider the Swift–Hohenberg equation (4.39), but instead of looking for solutions that are 2π-periodic in x, we seek solutions that satisfy the boundary conditions

$$u(\pm h, t) = \frac{\partial u}{\partial x}(\pm h, t) = 0 \tag{4.46}$$

on some interval $[-h, h]$. We assume that h is large enough, so that we regard this new problem as a “small” perturbation of the equation (4.39).

Replacing the spatial periodicity of the solutions by the boundary conditions (4.46) breaks the translational invariance $\mathbf{T}_a$, but does not break the symmetries $\mathbf{T}$ and $\mathbf{U}$, and $u = 0$ remains a solution of the new problem. As a consequence, the eigenvalues of the linear operator $\mathbf{L}_\mu$ are no longer double, and for $\mu = 0$ the former 0 eigenvalue splits into two simple, negative eigenvalues, which are close to 0, of order $O(1/h^3)$ as $h \to \infty$. The other eigenvalues are all negative and at least of order $O(1/h^2)$. It is then convenient to rescale the variables in order to push the eigenvalues of order $O(1/h^2)$ at a distance of order $O(1)$ from the imaginary axis. Then the two eigenvalues of order $O(1/h^3)$ are changed into eigenvalues of order $O(1/h)$, which allows us to use a center manifold reduction, as described in Remark 4.21, when the critical spectrum σ_0 does not lie on the imaginary axis, but stays close to it. In addition to the original parameter μ, we now have a second small parameter $\varepsilon = O(1/h)$, so that this case is indeed a small perturbation of the original problem.

Taking into account the fact that 0 is always a solution, and that in this new problem only the translational symmetry is broken, by arguing as in the previous cases one finds that the reduced equation is now modified at main orders as follows:

$$\frac{dA}{dt} = (\mu + a\varepsilon)A + b\varepsilon\bar{A} - 3A|A|^2,$$

where a and b are real coefficients. Using polar coordinates $A = re^{i\phi}$, we find the system

$$\begin{aligned} \frac{dr}{dt} &= r(\mu + a\varepsilon + b\varepsilon\cos 2\phi - 3r^2) \\ \frac{d\phi}{dt} &= -b\varepsilon\sin(2\phi). \end{aligned}$$

Steady solutions are found for $\phi \in \{0, \pi/2, \pi, 3\pi/2\}$. Note that changing $\phi \mapsto \phi + \pi$ is equivalent to changing $r \mapsto -r$, so that we can restrict to the two cases $\phi = 0$ and $\phi = \pi/2$. The case $\phi = 0$ leads to *symmetric solutions*, i.e., invariant under $\mathbf{U}$, since $A = \bar{A}$, whereas the case $\phi = \pi/2$ leads to *antisymmetric solutions*, since $\bar{A} = -A$. It turns out that *symmetric solutions* bifurcate for $\mu = -(a+b)\varepsilon$ and have the amplitude given by $r_S^2 = 1/3(\mu + (a+b)\varepsilon)$. Their stability is determined by the sign of the two eigenvalues $-6r_S^2$, $-2b\varepsilon$. *Antisymmetric solutions* bifurcate for $\mu = (b-a)\varepsilon$, and have the amplitude given by $r_A^2 = 1/3(\mu + (a-b)\varepsilon)$. Their stability is determined by the sign of the two eigenvalues $-6r_A^2$, $2b\varepsilon$. In particular, it follows that the *stabilities of these two branches of solutions are opposite* (see Figure 4.4 for a typical bifurcation diagram).

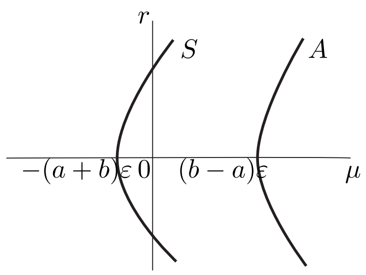


Figure 4.4: Bifurcation diagram for the Swift–Hohenberg equation with boundary conditions (4.46), for a fixed $\varepsilon = O(1/h)$. The two curves S and A represent the branches of symmetric and antisymmetric solutions, respectively.

Remark 4.41. *This question has a major physical importance for many hydrodynamic stability problems where, for a large aspect ratio apparatus, one replaces, for mathematical* *convenience, the physical boundary conditions by periodic boundary conditions (large periods), as for example in Section 6. On the model equation SHE, a complete mathematical justification of the new amplitude equation obtained for Dirichlet–Neumann boundary conditions, as a perturbation of the periodic case, can be found in [73], while this is still a mathematically open problem for classical hydrodynamic stability problems like the ones in Section 6.*

4.4.3 Elliptic PDE in a Strip

Consider the elliptic problem

$$\begin{aligned}&\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\nu v+g\left(v,\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)=0,\\ &v(x,0)=v(x,\pi)=0,\end{aligned}$$

where $v(x,y)\in\mathbb{R}$ for $(x,y)\in\mathbb{R}\times(0,\pi)$, ν is a real parameter, and we assume that $g\in C^k(\mathbb{R}^3,\mathbb{R})$, with $g(u,v,w)=O(|u|^2+|v|^2+|w|^2)$, as $(u,v,w)\to 0$. We further assume that g is even in its second argument.

Formulation and Symmetries This problem enters our setting when we take as our time variable the unbounded spatial variable $x\in\mathbb{R}$, and so write the problem in the form

$$\frac{du}{dx}=\mathbf{L}_\nu u+\mathbf{R}(u). \tag{4.47}$$

This formulation of the problem is also called "spatial dynamics" formulation. The idea of spatial dynamics goes back to [42] and was used for finding bounded solutions of elliptic PDEs in cylindrical domains.

We obtain the equation (4.47) by taking $u=(u_1,u_2)=(v,\partial v/\partial x)$, and

$$\mathbf{L}_\nu=\begin{pmatrix}0 & 1\\ -\frac{d^2}{dy^2}-\nu & 0\end{pmatrix},\quad \mathbf{R}(u)=\begin{pmatrix}0\\ -g\left(u_1,u_2,\frac{du_1}{dy}\right)\end{pmatrix}.$$

We choose the spaces

$$\mathcal{X}=H_0^1(0,\pi)\times L^2(0,\pi),\quad \mathcal{Z}=\left(H^2(0,\pi)\cap H_0^1(0,\pi)\right)\times H_0^1(0,\pi),$$

such that $\mathbf{L}_\nu$ is a closed operator in $\mathcal{X}$ with domain $\mathcal{Z}$, which contains the boundary conditions, and such that $\mathbf{R}(u)\in\mathcal{X}$ for $u\in\mathcal{Z}$, and $\mathbf{R}$ is of class $C^k(\mathcal{Z},\mathcal{X})$.

Notice that the elliptic equation is invariant under $(x,v)\mapsto(-x,v)$, since we assumed that g is even in its second argument. This induces a *reversibility symmetry* for (4.47), i.e., the vector field on the right hand side anticommutes with the symmetry $\mathbf{S}$ defined by

$$\mathbf{S}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}u_1\\ -u_2\end{pmatrix}.$$

As in the previous examples we next look at the spectrum of $\mathbf{L}_\nu$. We have again that $\mathcal{Z}$ is compactly embedded in $\mathcal{X}$, so $\mathbf{L}_\nu$ has a compact resolvent and its spectrum consists of isolated eigenvalues with finite multiplicities. The eigenvalue problem reads

$$\begin{aligned} u_2 &= \lambda u_1 \\ -u_1'' - \nu u_1 &= \lambda u_2, \end{aligned}$$

in which u_1 satisfies the boundary conditions $u_1(0) = u_1(\pi) = 0$. Then a direct computation shows that the spectrum σ of $\mathbf{L}_\nu$ is

$$\sigma = \{\lambda_n^\pm = \pm\sqrt{n^2 - \nu}\ ;\ n \in \mathbb{N}^*\}. \tag{4.48}$$

Notice that σ is symmetric with respect to both the imaginary and the real axis in the complex plane, due to the reversibility symmetry and the fact that $\mathbf{L}_\nu$ is a real operator. When $\nu \neq p^2$, for any integer p, the eigenvalues are all simple and real except for a finite number that are purely imaginary. When $\nu = p^2$ for some nonzero integer p, we find that 0 is a double eigenvalue, $\lambda_p^\pm = 0$, and the eigenvalues $\lambda_n^\pm$ with $n < p$ are purely imaginary, whereas the eigenvalues $\lambda_n^\pm$ with $n > p$ are real. Consequently, we can use the center manifold theorem, for values of ν close to $\nu_p = p^2$, for any $p \geq 1$.

Reduced System We focus here on values of ν close to $\nu_1 = 1$ and set $\nu = 1 + \mu$. We rewrite the equation (4.47) in the form

$$\frac{du}{dx} = \mathbf{L}u + \mathbf{R}(u, \mu),$$

with

$$\mathbf{L} = \mathbf{L}_1, \quad \mathbf{R}(u, \mu) = \mathbf{R}(u) + (\mathbf{L}_{1+\mu} - \mathbf{L}_1)u.$$

From the arguments above it is now easy to check that Hypotheses 4.16 and 4.14 hold. In Hypothesis 4.14 we have $\sigma_0 = \{0\}$ with 0 a geometrically simple and algebraically double eigenvalue. The corresponding spectral subspace $\mathcal{E}_0$ is spanned by

$$\zeta_0 = \begin{pmatrix} \sin y \\ 0 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} 0 \\ \sin y \end{pmatrix},$$

which satisfy $\mathbf{L}\zeta_0 = 0$ and $\mathbf{L}\zeta_1 = \zeta_0$, respectively. Also notice that

$$\mathbf{S}\zeta_0 = \zeta_0, \quad \mathbf{S}\zeta_1 = -\zeta_1.$$

Further, Hypothesis 4.6 can be checked as in the example in Section 4.4.1. In addition, the reversibility symmetry $\mathbf{S}$ satisfies Hypothesis 4.30.

We can now apply the results in Theorems 4.27 and 4.31 and obtain a family of two-dimensional center manifolds for μ sufficiently small. For solutions on the center manifold we write

$$u = A\zeta_0 + B\zeta_1 + \mathbf{\Psi}(A, B, \mu),$$

where $A(t) \in \mathbb{R}$, $B(t) \in \mathbb{R}$, and $\mathbf{\Psi}$ takes values in $\mathcal{Z}_h$. This leads to a reduced equation of the form

$$\begin{aligned} \frac{dA}{dx} &= f(A, B, \mu) \\ \frac{dB}{dx} &= g(A, B, \mu), \end{aligned} \tag{4.49}$$

in which the vector field (f,g) satisfies

$$(f,g)(0,0,\mu) = (0,0), \quad D(f,g)(0,0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In addition, the vector field is reversible, it anticommutes with the symmetry $\mathbf{S}_0$ induced by $\mathbf{S}$ acting through

$$\mathbf{S}_0(A,B) = (A,-B),$$

and the reduction function $\mathbf{\Psi}$ commutes with $\mathbf{S}$,

$$\mathbf{S\Psi}(A,B,\mu) = \mathbf{\Psi}(A,-B,\mu). \tag{4.50}$$

We come back to this example in section 5.4.3.

5 Normal forms

In this section we present a number of results from the theory of normal forms. The idea of normal forms consists in finding a polynomial change of variable which "improves" locally a nonlinear system, in order to more easily recognize its dynamics. As we shall see, normal form transformations apply to general classes of nonlinear systems in $\mathbb{R}^n$ near a fixed point, here the origin, by just assuming a certain smoothness of the vector field. In particular, this theory applies to the reduced systems provided by the center manifold theory given in Section 4.

5.1 Main Theorem

We consider a differential equation in $\mathbb{R}^n$ of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u), \tag{5.1}$$

in which $\mathbf{L}$ and $\mathbf{R}$ represent the linear and nonlinear terms, respectively. More precisely, we assume that the following holds.

Hypothesis 5.1. *Assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in (5.1) have the following properties:*

(i) $\mathbf{L}$ *is a linear map in* $\mathbb{R}^n$*;*

(ii) for some $k \geq 2$*, there exists a neighborhood* $\mathcal{V} \subset \mathbb{R}^n$ *of* 0 *such that* $\mathbf{R} \in \mathcal{C}^k(\mathcal{V}, \mathbb{R}^n)$ *and*

$$\mathbf{R}(0) = 0, \quad D\mathbf{R}(0) = 0.$$

Our purpose is to transform this system, in a neighborhood of the origin, in such a way that the Taylor expansion of the transformed nonlinear vector field contains a *minimal number of terms at every order*. The following result shows the existence of a polynomial change of variables leading to a transformed vector field, which, as we shall see later, has this property.

Theorem 5.2 (Normal form theorem)**.** *Consider the system (5.1) and assume that Hypothesis 5.1 holds. Then for any positive integer p, $2 \le p \le k$, there exists a polynomial $\mathbf{\Phi} : \mathbb{R}^n \to \mathbb{R}^n$ of degree p, with*

$$\mathbf{\Phi}(0) = 0, \quad D\mathbf{\Phi}(0) = 0,$$

and such that the change of variable

$$u = v + \mathbf{\Phi}(v) \tag{5.2}$$

defined in a neighborhood of the origin in $\mathbb{R}^n$ transforms the equation (5.1) into the "normal form"

$$\frac{dv}{dt} = \mathbf{L}v + \mathbf{N}(v) + \boldsymbol{\rho}(v), \tag{5.3}$$

with the following properties:

(i) $\mathbf{N} : \mathbb{R}^n \to \mathbb{R}^n$ *is a polynomial of degree p, satisfying*

$$\mathbf{N}(0) = 0, \quad D\mathbf{N}(0) = 0.$$

(ii) The equality

$$\mathbf{N}(e^{t\mathbf{L}^*}v) = e^{t\mathbf{L}^*}\mathbf{N}(v), \tag{5.4}$$

holds for all $(t, v) \in \mathbb{R} \times \mathbb{R}^n$, where $\mathbf{L}^*$ represents the adjoint of $\mathbf{L}$.

(iii) $\boldsymbol{\rho}$ *is a map of class $\mathcal{C}^k$ in a neighborhood of 0, such that*

$$\boldsymbol{\rho}(v) = o(\|v\|^p).$$

Remark 5.3 (Equivalent characterization of the normal form)**.** *Instead of the characterization (5.4) for the polynomial $\mathbf{N}$, it may be advantageous to use the following equivalent characterization*

$$D\mathbf{N}(v)\mathbf{L}^*v = \mathbf{L}^*\mathbf{N}(v) \text{ for all } v \in \mathbb{R}^n. \tag{5.5}$$

Indeed, the following identity is valid for any $(t, v) \in \mathbb{R} \times \mathbb{R}^n$:

$$\frac{d}{dt}\left(e^{-t\mathbf{L}^*}\mathbf{N}(e^{t\mathbf{L}^*}v)\right) = e^{-t\mathbf{L}^*}\left(-\mathbf{L}^*\mathbf{N}(e^{t\mathbf{L}^*}v) + D\mathbf{N}(e^{t\mathbf{L}^*}v)\mathbf{L}^*e^{t\mathbf{L}^*}v\right).$$

Consequently, if (5.4) holds, then the left hand side in the above equality vanishes, and by taking $t = 0$ in the right hand side we obtain (5.5). Conversely, writing (5.5) with $e^{t\mathbf{L}^*}v$ instead of v implies that $e^{-t\mathbf{L}^*}\mathbf{N}(e^{t\mathbf{L}^*}v)$ is independent of t, which gives (5.4).

Remark 5.4 (Uniqueness of the normal form)**.** *As we shall see from the proof of this theorem, the choice of the polynomial $\mathbf{N}$ is not unique. Actually, one can add to the polynomial $\mathbf{N}$ satisfying one of the equivalent characterizations (5.4) or (5.5) any polynomial $\mathbf{Q}$ which belongs to the range of the linear operator $\mathcal{A}_{\mathbf{L}}$ acting on the space of polynomials $\mathbf{\Phi} : \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$(\mathcal{A}_{\mathbf{L}}\mathbf{\Phi})(v) = D\mathbf{\Phi}(v)\mathbf{L}v - \mathbf{L}\mathbf{\Phi}(v) \text{ for all } v \in \mathbb{R}^n.$$

Of course the new polynomial $\mathbf{N}+\mathbf{Q}$ does not satisfy (5.4) and (5.5) anymore, but the change of variables $\mathbf{\Phi}$ still exists. This property may sometimes allow one to further simplify the normal form (e.g., see Remark 5.10).

Remark 5.5. *In applications we often use the characterizations (5.4) or (5.5) in a complex basis in which* $\mathbf{L}^*$ *is diagonal, or triangular (Jordan form). The formulations of (5.4) and (5.5) are valid in such a basis, as well. Indeed, denote by* $\mathbf{P}$ *the matrix for a change of basis, which may be complex, such that*

$$\mathbf{P}^{-1}\mathbf{L}^*\mathbf{P} = \mathbf{T}^*.$$

Replacing $v = \mathbf{P}w$ *into (5.5) we find*

$$D_v\mathbf{P}^{-1}\mathbf{N}(\mathbf{P}w)\mathbf{P}\mathbf{T}^*w = \mathbf{T}^*\mathbf{P}^{-1}\mathbf{N}(\mathbf{P}w).$$

Consequently, the polynomial $\widetilde{\mathbf{N}}$ *defined through*

$$\widetilde{\mathbf{N}}(w) \stackrel{def}{=} \mathbf{P}^{-1}\mathbf{N}(\mathbf{P}w)$$

satisfies

$$D_w\widetilde{\mathbf{N}}(w)\mathbf{T}^*w = \mathbf{T}^*\widetilde{\mathbf{N}}(w),$$

which is equivalent to (5.5).

Remark 5.6. *(i) Theorem 5.2 has been proved in [14] in its elementary formulation, given below in Subsection 5.1.1. The characterization (5.5) is in fact contained (a little hidden) in the general work of Belitskii [3], using more sophisticated methods of algebraic geometry.*

(ii) Some normal form results are also available in infinite- dimensional spaces for very specific problems, but there is no general result in this situation. The result in Theorem 5.2 suffices for the analysis of the reduced systems obtained by a center manifold reduction, since these are all finite-dimensional.

5.1.1 Proof of Theorem 5.2

We give in this section the proof of the normal form Theorem 5.2.

Consider the Taylor expansion of $\mathbf{R}$,

$$\mathbf{R}(u) = \sum_{2\le q\le p} \mathbf{R}_q(u^{(q)}) + o(\|u\|^p)$$

for a given p, $2 \le p \le k$, where $u^{(q)} = (u, \dots, u) \in (\mathbb{R}^n)^q$, with $u \in \mathbb{R}^n$ repeated q times, and $\mathbf{R}_q$ is the q-linear symmetric map on $(\mathbb{R}^n)^q$ given through

$$\mathbf{R}_q(u^{(q)}) = \frac{1}{q!}D^q\mathbf{R}(0)(u^{(q)}).$$

Similarly, we write the polynomials $\mathbf{\Phi}$ and $\mathbf{N}$ in the form

$$\mathbf{\Phi}(v) = \sum_{2\le q\le p} \mathbf{\Phi}_q(v^{(q)}), \quad \mathbf{N}(v) = \sum_{2\le q\le p} \mathbf{N}_q(v^{(q)}),$$

with $\mathbf{\Phi}_q$ and $\mathbf{N}_q$ q-linear symmetric maps on $(\mathbb{R}^n)^q$.

Differentiating (5.2) with respect to t and replacing du/dt and dv/dt from (5.1) and (5.3), respectively, leads to the identity

$$(\mathbb{I} + D\mathbf{\Phi}(v))\,(\mathbf{L}v + \mathbf{N}(v) + \boldsymbol{\rho}(v)) = \mathbf{L}(v + \mathbf{\Phi}(v)) + \mathbf{R}(v + \mathbf{\Phi}(v)), \tag{5.6}$$

which should be valid for all v in a neighborhood of 0. Our purpose it to determine $\mathbf{\Phi}$ and $\mathbf{N}$ from this equality. By identifying the Taylor expansions on both sides, we obtain at order 2

$$D\mathbf{\Phi}_2(v^{(2)})\mathbf{L}v - \mathbf{L}\mathbf{\Phi}_2(v^{(2)}) = \mathbf{R}_2(v^{(2)}) - \mathbf{N}_2(v^{(2)}), \tag{5.7}$$

and then at any order q, $3 \leq q \leq p$, we have

$$D\mathbf{\Phi}_q(v^{(q)})\mathbf{L}v - \mathbf{L}\mathbf{\Phi}_q(v^{(q)}) = \mathbf{Q}_q(v^{(q)}) - \mathbf{N}_q(v^{(q)}), \tag{5.8}$$

with

$$\begin{aligned}\mathbf{Q}_q(v^{(q)}) \quad &= \quad - \sum_{2\leq r\leq q-1} D\mathbf{\Phi}_r(v^{(r)})\mathbf{N}_{q-r+1}(v^{(q-r+1)}) + \\ &\quad + \sum_{r_1+\cdots+r_\ell=q, \quad r_j\geq 1} \mathbf{R}_\ell\left(\mathbf{\Phi}_{r_1}(v^{(r_1)}), \mathbf{\Phi}_{r_2}(v^{(r_2)}), \ldots, \mathbf{\Phi}_{r_\ell}(v^{(r_\ell)})\right),\end{aligned}$$

where we have set $\mathbf{\Phi}_1(v) = v$. Notice that if $\mathbf{\Phi}_l$ and $\mathbf{N}_l$ are known for any l, $2 \leq l \leq q-1$, then $\mathbf{Q}_q$ is known. Therefore, we can determine $\mathbf{\Phi}$ and $\mathbf{N}$ by successively finding $(\mathbf{\Phi}_2, \mathbf{N}_2)$, $(\mathbf{\Phi}_3, \mathbf{N}_3)$, and so on, from (5.7) and (5.8).

The equations (5.7) and (5.8) have the same structure; more precisely, they are both of the form

$$\mathcal{A}_\mathbf{L}\mathbf{\Phi}_q = \mathbf{Q}_q - \mathbf{N}_q, \tag{5.9}$$

in which $\mathcal{A}_\mathbf{L}$ is a linear map (also called "homological operator") acting on the space of polynomials $\mathbf{\Phi} : \mathbb{R}^n \to \mathbb{R}^n$ through

$$(\mathcal{A}_\mathbf{L}\mathbf{\Phi})(v) = D\mathbf{\Phi}(v)\mathbf{L}v - \mathbf{L}\mathbf{\Phi}(v). \tag{5.10}$$

A key property of $\mathcal{A}_\mathbf{L}$ is that it leaves invariant the subspace $\mathcal{H}_q$ of homogeneous polynomials of degree q, for any positive integer q. In the equality (5.9), $\mathbf{Q}_q$ is known, and we have to determine $\mathbf{\Phi}_q$ and $\mathbf{N}_q$. It is clear that if $\mathcal{A}_\mathbf{L}|_{\mathcal{H}_q}$ is invertible, then we can take $\mathbf{N}_q = 0$, which gives the simplest solution here. However, this is not always the case, and the condition for solving (5.9) is that $\mathbf{Q}_q - \mathbf{N}_q$ lies in the range of the operator $\mathcal{A}_\mathbf{L}$. We claim that this condition is achieved when (5.4), or equivalently (5.5), is satisfied by $\mathbf{N}_q$.

Indeed, we define below a scalar product in the space $\mathcal{H}$ of polynomials of degree p, such that the adjoint operator $(\mathcal{A}_\mathbf{L})^*$ of $\mathcal{A}_\mathbf{L}$ with respect to this scalar product is $\mathcal{A}_{\mathbf{L}^*}$, where $\mathbf{L}^*$ is the adjoint of $\mathbf{L}$ with respect to the canonical Euclidean scalar product in $\mathbb{R}^n$. Then $\mathbf{Q}_q - \mathbf{N}_q$ belongs to the range of $\mathcal{A}_\mathbf{L}$ if

$$\mathbf{Q}_q - \mathbf{N}_q \in \ker(\mathcal{A}_{\mathbf{L}^*})^\perp = im(\mathcal{A}_\mathbf{L}),$$

or, equivalently,

$$\mathbf{P}_{\ker(\mathcal{A}_{\mathbf{L}^*})}(\mathbf{Q}_q - \mathbf{N}_q) = 0,$$

where $\mathbf{P}_{\ker(\mathcal{A}_{\mathbf{L}^*})}$ is the orthogonal projection on $\ker(\mathcal{A}_{\mathbf{L}^*})$ in the space $\mathcal{H}$ of polynomials of degree p. It is then natural to choose

$$\mathbf{N}_q = \mathbf{P}_{\ker(\mathcal{A}_{\mathbf{L}^*})}\mathbf{Q}_q.$$

Of course, this choice is not unique, since we can add to $\mathbf{N}_q$ any term in the range of $\mathcal{A}_{\mathbf{L}}$ (this then implies Remark 5.4). Furthermore, we shall see that the projection $\mathbf{P}_{\ker(\mathcal{A}_{\mathbf{L}^*})}$ leaves invariant the subspace $\mathcal{H}_q$, so that $\mathbf{N}_q \in \ker \mathcal{A}_{\mathbf{L}^*}|_{\mathcal{H}_q}$. In particular, this shows that (5.5) holds for $\mathbf{N}_q$. With this choice for $\mathbf{N}_q$, we can now solve (5.9) and obtain a solution $\mathbf{\Phi}_q$, which is determined up to an arbitrary element in the kernel of $\mathcal{A}_{\mathbf{L}}$. A possible, but not unique, choice is to choose the unique solution $\mathbf{\Phi}_q$ orthogonal to $\ker \mathcal{A}_{\mathbf{L}}$ in $\mathcal{H}_q$. Summarizing, this shows that (5.9) possesses a solution $(\mathbf{\Phi}_q, \mathbf{N}_q)$ with $\mathbf{N}_q$ satisfying (5.5). Solving successively for $q = 2, \dots, p$, we obtain the polynomials $\mathbf{\Phi}$ and $\mathbf{N}$ in the theorem, with $\mathbf{N}$ satisfying (5.5).

To finish the proof, it remains to define the scalar product in the space $\mathcal{H}$ such that

$$(\mathcal{A}_{\mathbf{L}})^* = \mathcal{A}_{\mathbf{L}^*}, \tag{5.11}$$

and to check that the orthogonal projection $\mathbf{P}_{\ker(\mathcal{A}_{\mathbf{L}^*})}$ on $\ker(\mathcal{A}_{\mathbf{L}^*})$ leaves invariant the subspace $\mathcal{H}_q$.

For a pair of scalar polynomials $P, P' : \mathbb{R}^n \to \mathbb{R}$ we define

$$\langle P|P'\rangle \stackrel{\text{def}}{=} P(\partial_u)P'(u)|_{u=0}, \tag{5.12}$$

where $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ and $\partial_u = (\partial/\partial u_1, \dots, \partial/\partial u_n)$. The equality (5.12) defines a scalar product in the linear space of scalar polynomials $P : \mathbb{R}^n \to \mathbb{R}$. To see this, it is sufficient to take the canonical basis of the space of scalar polynomials, consisting of monomials $u_1^{\alpha_1} \dots u_n^{\alpha_n}$, and to check that

$$\langle u_1^{\alpha_1} \dots u_n^{\alpha_n} | u_1^{\beta_1} \dots u_n^{\beta_n}\rangle = \alpha_1! \dots \alpha_n! \delta_{\alpha_1\beta_1} \dots \delta_{\alpha_n\beta_n},$$

where $\delta_{\alpha_j\beta_j} = 1$ if $\alpha_j = \beta_j$, and $\delta_{\alpha_j\beta_j} = 0$ otherwise. (Notice that this scalar product can be extended to complex-valued polynomials $P : \mathbb{C}^n \to \mathbb{C}$ by taking

$$\langle P|P'\rangle \stackrel{\text{def}}{=} P(\partial_u)\overline{P}'(u)|_{u=0},$$

where $\overline{P}(u) \stackrel{\text{def}}{=} \overline{P(\overline{u})}$.)

Now we define a scalar product on $\mathcal{H}$ by taking

$$\langle \mathbf{\Phi}|\mathbf{\Phi}'\rangle = \sum_{j=1}^{n} \langle \Phi_j|\Phi_j'\rangle$$

for $\mathbf{\Phi} = (\Phi_1, \dots, \Phi_n) \in \mathcal{H}$, $\mathbf{\Phi}' = (\Phi_1', \dots, \Phi_n') \in \mathcal{H}$. An important property of this scalar product (used in theoretical physics) is that the adjoint of the multiplication by u_j is the differentiation with respect to u_j,

$$\langle u_j P|P'\rangle = \partial_{u_j} P(\partial_u)P'(u)|_{u=0} = P(\partial_u)\partial_{u_j} P'(u)|_{u=0} = \langle P|\partial_{u_j} P'\rangle.$$

For our purpose, the most interesting property is the equality

$$\langle P\circ \mathbf{T}|P'\rangle = \langle P|P'\circ \mathbf{T}^*\rangle, \tag{5.13}$$

in which $\mathbf{T}$ is any invertible linear map, and $\mathbf{T}^*$ is the adjoint of $\mathbf{T}$ with respect to the Euclidean scalar product in $\mathbb{R}^n$. To show (5.13), consider the change of variable $u = \mathbf{T}^* v$, which means

$$u_i = \sum_{j=1}^{n} T_{ji} v_j,$$

for $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ and $\mathbf{T} = (T_{ij})_{1\le i,j\le n}$. Then

$$\frac{\partial u_i}{\partial v_j} = T_{ji}, \quad \frac{\partial}{\partial v_j} = \sum_{i=1}^{n} T_{ji}\frac{\partial}{\partial u_i},$$

so that $\partial_v = \mathbf{T}\partial_u$. Using this equality and the fact that $u = 0$ is equivalent to $v = 0$, we find

$$\langle P\circ \mathbf{T}|P'\rangle = P(\mathbf{T}\partial_u)P'(u)|_{u=0} = P(\partial_v)P'(\mathbf{T}^* v)|_{v=0} = \langle P|P'\circ \mathbf{T}^*\rangle,$$

which proves (5.13).

We use the identity (5.13) to determine the adjoint of $\mathcal{A}_{\mathbf{L}}$. We take $\mathbf{T} = e^{-t\mathbf{L}}$, for which we find $\mathbf{T}^* = e^{-t\mathbf{L}^*}$ and $\mathbf{T}^{-1} = e^{t\mathbf{L}}$. Then from (5.13) we obtain

$$\langle e^{-t\mathbf{L}}\mathbf{\Phi}\circ e^{t\mathbf{L}}|\mathbf{\Phi'}\rangle = \langle \mathbf{\Phi}|e^{-t\mathbf{L}^*}\mathbf{\Phi'}\circ e^{t\mathbf{L}^*}\rangle$$

for any $\mathbf{\Phi}, \mathbf{\Phi'} \in \mathcal{H}$. Differentiating this equality with respect to t at $t = 0$, leads to

$$\langle \mathcal{A}_{\mathbf{L}}\mathbf{\Phi}|\mathbf{\Phi'}\rangle = \langle \mathbf{\Phi}|\mathcal{A}_{\mathbf{L}^*}\mathbf{\Phi'}\rangle.$$

This proves the formula for the adjoint (5.11).

Finally, the identity above also holds in the subspaces $\mathcal{H}_q$ of homogeneous polynomials of degree q, which are all invariant under the actions of both $\mathcal{A}_{\mathbf{L}}$ and $\mathcal{A}_{\mathbf{L}^*}$. Consequently,

$$\ker(\mathcal{A}_{\mathbf{L}^*}|_{\mathcal{H}_p}) = \ker \mathcal{A}_{\mathbf{L}^*} \cap \mathcal{H}_p,$$

and since monomials with different degrees are orthogonal to each other, this implies the invariance of $\mathcal{H}_p$ under the orthogonal projection $\mathbf{P}_{\ker \mathcal{A}_{\mathbf{L}^*}}$. This ends the proof of Theorem 5.2.

In the next subsections, we apply this theorem to different cases in dimensions 2, 3, and 4. In all these cases the linear map $\mathbf{L}$ has purely imaginary eigenvalues, only, just as the linear part has in the reduced systems obtained from the center manifold reduction.

5.1.2 Examples in Dimension 2: $i\omega$, 0^2

We discuss in this section two cases in dimension 2: $i\omega$, where $\mathbf{L}$ has a pair of simple complex eigenvalues $\pm i\omega$, and 0^2, where $\mathbf{L}$ has a double zero eigenvalue with a Jordan block of length 2.

The case $i\omega$ corresponds to a matrix

$$\mathbf{L} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix},$$

where $\omega > 0$, and $\mathbf{L}$ has the simple eigenvalues $\pm i\omega$. In this situation it is more convenient to identify $\mathbb{R}^2$ with the diagonal $\{(z, \bar{z}); z \in \mathbb{C}\}$ in $\mathbb{C}^2$ and to choose a complex basis of eigenvectors $\{\zeta, \overline{\zeta}\}$ with $\zeta = (1, -i)$, such that $\mathbf{L}$ becomes

$$\mathbf{L} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}. \tag{5.14}$$

A vector in $\mathbb{R}^2$ is now represented as

$$u = A\zeta + \overline{A\zeta}, \quad A \in \mathbb{C}.$$

Applying Theorem 5.2, we now prove the following result.

Lemma 5.7 ($i\omega$ normal form)**.** *Assume that the 2×2-matrix $\mathbf{L}$ takes the form (5.14) in a complex basis $\{\zeta, \bar{\zeta}\}$, in which a vector $u \in \mathbb{R}^2$ is represented by $u = (A, \overline{A})$, with $A \in \mathbb{C}$. Then the polynomial $\mathbf{N}$ in Theorem 5.2 is of the form*

$$\mathbf{N}(u) = (AQ(|A|^2), \overline{AQ}(|A|^2)),$$

where Q is a complex-valued polynomial in its argument, satisfying $Q(0) = 0$.

Proof. In order to determine the normal form in this case, it is convenient to use the identity (5.4) and Remark 5.5. We have

$$e^{t\mathbf{L}^*} = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix},$$

and denoting $\mathbf{N} = (P(A, \overline{A}), \overline{P}(A, \overline{A}))$, from (5.4) we obtain that

$$P(e^{-i\omega t}A, e^{i\omega t}\overline{A}) = e^{-i\omega t}P(A, \overline{A}).$$

In particular, this shows that the normal form in this case commutes with all rotations in the complex plane (with this choice of the basis). Using Lemma 3.4, we find that

$$P(A, \overline{A}) = AQ(|A|^2),$$

where Q is a complex-valued polynomial in its argument. Moreover, $Q(0) = 0$ since $D\mathbf{N}(0) = 0$, which completes the proof. ∎

Exercise 5.8. *Compute the terms up to order* 2 *in the normal of the system* (3.31)*, with* $\mu = 0$.
Hint: *Redo the calculations in Section 3.2 with* $\mu = 0$.

Next we consider the case 0^2 where $\mathbf{L}$ has a double zero eigenvalue with a Jordan block of length 2.

Lemma 5.9 (0^2 normal form)**.** *Assume that the matrix* $\mathbf{L}$ *is in Jordan form*

$$\mathbf{L} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

in a basis of $\mathbb{R}^2$ *in which a vector* $u \in \mathbb{R}^2$ *is represented by* $u = (A, B) \in \mathbb{R}^2$*. Then the polynomial* $\mathbf{N}$ *in Theorem 5.2 is of the form*

$$\mathbf{N}(u) = (AP(A),\ BP(A) + Q(A)),$$

where P *and* Q *are real-valued polynomials, satisfying* $P(0) = Q(0) = Q'(0) = 0$.

Proof. We set

$$\mathbf{N}(u) = (\Phi_1(A, B), \Phi_2(A, B)),$$

where Φ_1 and Φ_2 are polynomials in (A, B). Then we have $\mathbf{L}^*(A, B) = (0, A)$ and using the identity (5.5) we obtain

$$A\frac{\partial \Phi_1}{\partial B} = 0, \quad A\frac{\partial \Phi_2}{\partial B} = \Phi_1.$$

Consequently, Φ_1 does not depend upon B, $\Phi_1(A, B) = \phi_1(A)$, and since the polynomial $A\partial\Phi_2/\partial B = \Phi_1$ is divisible by A, there exists a polynomial P such that

$$\Phi_1(A, B) = AP(A).$$

Then the equation for the polynomial Φ_2 leads to

$$\Phi_2(A, B) = BP(A) + Q(A),$$

with Q a polynomial. Finally, we find that $P(0) = Q(0) = Q'(0) = 0$, since $\mathbf{N}(0) = 0$ and $D\mathbf{N}(0) = 0$. ∎

Remark 5.10. *(i) Notice that the kernel of the operator* $\mathcal{A}_{\mathbf{L}^*}$ *in the proof of Theorem 5.2 in the space* $\mathcal{H}_p$ *of homogeneous polynomials of degree* q *is in this case two-dimensional, spanned by*

$$(A^q, BA^{q-1}), \quad (0, A^q).$$

Furthermore, $(-A^q, qBA^{q-1})$ *is orthogonal to this two-dimensional space, so that it belongs to the range of* $\mathcal{A}_{\mathbf{L}}$*. As it was noticed in Remark 5.4, we can add to* $\mathbf{N}$ *any term in the range of* $\mathcal{A}_{\mathbf{L}}$*. In particular, in this case we can then choose* $\mathbf{N}$ *such that its first component is* 0*, which gives a simpler normal form,*

$$\mathbf{N}(u) = (0, BP_1(A) + Q_1(A)),$$

where P_1 *and* Q_1 *are polynomials such that* $P_1(0) = Q_1(0) = Q_1'(0) = 0$.

(ii) Alternatively, we can obtain this simpler normal form starting from the result in Lemma 5.9, which gives the system

$$\begin{aligned}\frac{dA}{dt} &= B + AP(A) + \rho_0(A,B) \\ \frac{dB}{dt} &= BP(A) + Q(A) + \rho_1(A,B),\end{aligned} \tag{5.15}$$

by making the change of variables

$$\widetilde{B} = B + AP(A) + \rho_0(A,B). \tag{5.16}$$

By the implicit function theorem, this change of variables is invertible:

$$B = \widetilde{B} - AP(A) + \widetilde{\rho}_0(A,\widetilde{B}),$$

and leads to the system

$$\begin{aligned}\frac{dA}{dt} &= \widetilde{B} \\ \frac{d\widetilde{B}}{dt} &= \widetilde{B}P_1(A) + Q_1(A) + \widetilde{\rho}_1(A,\widetilde{B}),\end{aligned}$$

with

$$P_1(A) = P(A) + \frac{d}{dA}(AP(A)), \quad Q_1(A) = Q(A) - A(P(A))^2.$$

Notice that in contrast to the result in the first part of this remark, in the first equation of the system above there is no longer a remainder. In turn, when going back to the change of variables from $(A,\widetilde{B})$ to u, this transformation is now not a polynomial.

Example: Computation of a 0^2 Normal Form

Consider the following second order differential equation

$$u'' = \alpha u^2 + \beta uu' + \gamma(u')^2, \tag{5.17}$$

with α, β, and γ real numbers.

Normal Form We set $U = (u,v)$, so that the equation takes the form

$$\frac{dU}{dt} = \mathbf{L}U + \mathbf{R}_2(U,U), \tag{5.18}$$

with

$$\mathbf{L} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{R}_2(U,\widetilde{U}) = \begin{pmatrix} 0 \\ \alpha u\widetilde{u} + \frac{\beta}{2}(u\widetilde{v} + \widetilde{u}v) + \gamma v\widetilde{v} \end{pmatrix}.$$

We are interested in computing the normal form of this system up to terms of order 2. Therefore it is enough to use the result in the normal form Theorem 5.2 with $p = 2$, i.e., to take the polynomial $\mathbf{\Phi}$ of the form

$$\mathbf{\Phi}(A,B) = A^2\mathbf{\Phi}_{20} + AB\mathbf{\Phi}_{11} + B^2\mathbf{\Phi}_{02}.$$

Then, according to Lemma 5.9 and Remark 5.10(ii), the change of variables

$$U = A\zeta_0 + B\zeta_1 + \mathbf{\Phi}(A,B), \tag{5.19}$$

where

$$\zeta_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

transforms system (5.18) into the normal form

$$\begin{aligned} \frac{dA}{dt} &= B \\ \frac{dB}{dt} &= aA^2 + bAB + O(|A|+|B|)^3, \end{aligned} \tag{5.20}$$

where a and b are real constants.

Computation of the Coefficients a ***and*** b In order to compute the coefficients a and b in this normal form we proceed as in the computation of the Hopf bifurcation in Section 3 (see Subsections 3.1 and 3.2).

First, substituting the change of variables (5.19) into the system (5.18) we find the equation

$$\begin{aligned} &\frac{dA}{dt}\zeta_0 + \frac{dB}{dt}\zeta_1 + \partial_A \mathbf{\Phi}(A,B)\frac{dA}{dt} + \partial_B \mathbf{\Phi}(A,B)\frac{dB}{dt} \\ &\qquad = B\zeta_0 + \mathbf{L\Phi} + \mathbf{R}_2(A\zeta_0 + B\zeta_1 + \mathbf{\Phi}, A\zeta_0 + B\zeta_1 + \mathbf{\Phi}), \end{aligned}$$

where we have used the fact that $\mathbf{L}\zeta_0 = 0$ and $\mathbf{L}\zeta_1 = \zeta_0$. Next, we substitute the expressions of dA/dt and dB/dt from (5.20) in the left hand side of the equality above. In the resulting equality we identify the monomials A^2, AB, B^2, and find that

$$a\zeta_1 = \mathbf{L\Phi}_{20} + \mathbf{R}_2(\zeta_0, \zeta_0), \tag{5.21}$$

$$b\zeta_1 + 2\mathbf{\Phi}_{20} = \mathbf{L\Phi}_{11} + 2\mathbf{R}_2(\zeta_0, \zeta_1), \tag{5.22}$$

$$\mathbf{\Phi}_{11} = \mathbf{L\Phi}_{02} + \mathbf{R}_2(\zeta_1, \zeta_1), \tag{5.23}$$

where

$$\mathbf{R}_2(\zeta_0,\zeta_0) = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \quad 2\mathbf{R}_2(\zeta_0,\zeta_1) = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \quad \mathbf{R}_2(\zeta_1,\zeta_1) = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}.$$

Each of equations (5.21)–(5.23) are nonhomogeneous linear systems of the form

$$\mathbf{L}\Phi = R, \quad \Phi, R \in \mathbb{R}^2,$$

which are not uniquely solvable, since $\mathbf{L}$ is not invertible. Notice that the range $im\mathbf{L}$ of $\mathbf{L}$ is given by $im\mathbf{L} = \{(u,0); u \in \mathbb{R}\} \subset \mathbb{R}^2$ and that the kernel $\ker \mathbf{L}$ is spanned by ζ_0. Consequently, the system $\mathbf{L}\Phi = R$ has a solution if and only if $R \in im\mathbf{L}$ and this solution is unique up to an element in $\ker \mathbf{L}$.

For the equation (5.21) we find

$$a\zeta_1 - \mathbf{R}_2(\zeta_0, \zeta_0) = \begin{pmatrix} 0 \\ a - \alpha \end{pmatrix},$$

so that the solvability condition $a\zeta_1 - \mathbf{R}_2(\zeta_0, \zeta_0) \in im\mathbf{L}$ is satisfied when

$$a = \alpha,$$

which determines the coefficient a in the normal form. Then the solution $\mathbf{\Phi}_{20}$ is any element of the kernel of $\mathbf{L}$,

$$\mathbf{\Phi}_{20} = \phi_{20}\zeta_0, \quad \phi_{20} \in \mathbb{R}.$$

Next, for equation (5.22) we have

$$b\zeta_1 + 2\mathbf{\Phi}_{20} - 2\mathbf{R}_2(\zeta_0, \zeta_1) = \begin{pmatrix} 2\phi_{20} \\ b - \beta \end{pmatrix},$$

so that the solvability condition for this equation determines the coefficient b, namely,

$$b = \beta.$$

This completes the calculation of the coefficients a and b.

Notice that it is not necessary to compute the solution $\mathbf{\Phi}_{11}$ of the equation (5.22) and to solve the equation (5.23), unless one needs to also compute the polynomial $\mathbf{\Phi}$ in the change of variables. Here we find

$$\mathbf{\Phi}_{11} = 2\phi_{20}\zeta_1 + \phi_{11}\zeta_0, \quad 2\phi_{20} = \gamma, \quad \mathbf{\Phi}_{02} = \phi_{11}\zeta_1 + \phi_{02}\zeta_0,$$

where the second equality is the solvability condition for the equation (5.23). In particular, this uniquely determines ϕ_{20}, whereas ϕ_{11} and ϕ_{02} are arbitrary. We can choose, for instance, $\phi_{11} = \phi_{02} = 0$, which then leads to the formula for $\mathbf{\Phi}$:

$$\mathbf{\Phi}(A, B) = \frac{\gamma}{2}A^2\zeta_0 + \gamma AB\zeta_1.$$

Remark 5.11. *In this example it was easy to determine the range $im\mathbf{L}$ of $\mathbf{L}$, and so to obtain the solvability conditions for the equations* (5.21)*–*(5.23)*. In general, a convenient way of finding these solvability conditions is with the help of the adjoint $\mathbf{L}^*$, since the kernel of the adjoint $\mathbf{L}^*$ is orthogonal to the range of $\mathbf{L}$. This means that the solvability conditions are orthogonality conditions on the kernel of the adjoint $\mathbf{L}^*$.*

5.1.3 Examples in Dimension 3: $0(i\omega)$, 0^3

We present in this section two cases in dimension 3: $0(i\omega)$, where $\mathbf{L}$ has a pair of simple complex eigenvalues $\pm i\omega$ and a simple eigenvalue at 0, and 0^3, where $\mathbf{L}$ has a triple zero eigenvalue with a Jordan block of length 3.

Lemma 5.12 ($0(i\omega)$ normal form). *Assume that the matrix* $\mathbf{L}$ *is of the form*

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix}$$

for some $\omega > 0$, *in a basis of* $\mathbb{R}^3$ *in which a vector* $u \in \mathbb{R}^3$ *is represented by* $u = (A, B, \overline{B})$, *with* $A \in \mathbb{R}$ *and* $B \in \mathbb{C}$. *Then the polynomial* $\mathbf{N}$ *in Theorem 5.2 is of the form*

$$\mathbf{N}(u) = (P(A, |B|^2),\ BQ(A, |B|^2),\ \overline{BQ}(A, |B|^2)),$$

where P *and* Q *are polynomials in their arguments, taking values in* $\mathbb{R}$ *and* $\mathbb{C}$, *respectively, and satisfying* $P(0,0) = \partial P/\partial A(0,0) = Q(0,0) = 0$.

Proof. We set

$$\mathbf{N}(u) = (P_0(A, B, \overline{B}), Q_0(A, B, \overline{B}), \overline{Q_0}(A, B, \overline{B})).$$

Then identity (5.4) leads to

$$\begin{aligned} P_0(A, e^{-i\omega t}B, e^{i\omega t}\overline{B}) &= P_0(A, B, \overline{B}), \\ Q_0(A, e^{-i\omega t}B, e^{i\omega t}\overline{B}) &= e^{-i\omega t}Q_0(A, B, \overline{B}), \end{aligned}$$

which holds for all $t \in \mathbb{R}$ and all $(A, B, \overline{B}) \in \mathbb{R} \times \mathbb{C}^2$. First, the same arguments as in the proof of Lemma 3.4, give the form of the dependency of Q_0 upon B, namely,

$$Q_0(A, B, \overline{B}) = BQ(A, |B|^2).$$

Since Q_0 is a polynomial in $(A, B, \overline{B})$ with $Q_0(0,0,0) = 0$ and $DQ_0(0,0,0) = 0$, we conclude that Q is a polynomial in its arguments with $Q(0,0) = 0$. Next, for the polynomial P_0 we take successively $\omega t = \arg B$ and $\omega t = \pi$, which give that

$$P_0(A, B, \overline{B}) = P_0(A, |B|, |B|) = P_0(A, -B, -\overline{B}).$$

Consequently, P_0 is of the form

$$P_0(A, B, \overline{B}) = P(A, |B|^2),$$

where P is a polynomial in its arguments and satisfies $P(0,0) = \partial P/\partial A(0,0) = 0$. ∎

In the case 0^3, it can be proved that the following result holds.

Lemma 5.13 (0^3 normal form). *Assume that the matrix* $\mathbf{L}$ *is in Jordan form*

$$\mathbf{L} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

in a basis of $\mathbb{R}^3$ *in which a vector* $u \in \mathbb{R}^3$ *is represented by* $u = (A, B, C)$, *with* $A, B, C \in \mathbb{R}$. *Then the polynomial* $\mathbf{N}$ *in Theorem 5.2 is of the form*

$$\mathbf{N}(u) = (AP_1(A, \widetilde{B}),\ BP_1(A, \widetilde{B}) + AP_2(A, \widetilde{B}),\ CP_1(A, \widetilde{B}) + BP_2(A, \widetilde{B}) + P_3(A, \widetilde{B})),$$

where

$$\widetilde{B} = B^2 - 2AC,$$

and P_1*,* P_2*, and* P_3 *are real-valued polynomials such that* $P_1(0,0) = P_2(0,0) = P_3(0,0) = \partial P_3/\partial A(0,0) = 0$*.*

Remark 5.14. *As in the case* 0^2*, we can use here Remark 5.4 and choose* $\mathbf{N}$ *such that its two first components vanish, i.e.,*

$$\mathbf{N}(u) = (0,\, 0,\, CP_1(A,\widetilde{B}) + BP_2(A,\widetilde{B}) + P_3(A,\widetilde{B})),$$

where the polynomials P_1*,* P_2*, and* P_3 *are real-valued such that* $P_1(0,0) = P_2(0,0) = P_3(0,0) = \partial P_3/\partial A(0,0) = 0$*.*

5.1.4 Examples in Dimension 4: $(i\omega_1)(i\omega_2)$, $(i\omega)^2$, $0^2(i\omega)$.

In this section we consider four cases of matrices $\mathbf{L}$ in $\mathbb{R}^4$. The first case is that in which $\mathbf{L}$ has two pairs of simple purely imaginary eigenvalues, $\pm i\omega_1$ and $\pm i\omega_2$.

Lemma 5.15 ($(i\omega_1)(i\omega_2)$ normal form)**.** *Assume that the matrix* $\mathbf{L}$ *is of the form*

$$\mathbf{L} = \begin{pmatrix} i\omega_1 & 0 & 0 & 0 \\ 0 & i\omega_2 & 0 & 0 \\ 0 & 0 & -i\omega_1 & 0 \\ 0 & 0 & 0 & -i\omega_2 \end{pmatrix},$$

where $\omega_1 \neq \omega_2$ *are positive real numbers, in a basis of* $\mathbb{R}^4$ *in which a vector* $u \in \mathbb{R}^4$ *is represented by* $u = (A, B, \overline{A}, \overline{B})$*, with* $A, B \in \mathbb{C}$*.*

(i) Assume that $\omega_1/\omega_2 \notin \mathbb{Q}$*. Then the polynomial* $\mathbf{N}$ *in Theorem 5.2 is of the form*

$$\mathbf{N}(u) = (AP(|A|^2,|B|^2),\, BQ(|A|^2,|B|^2),\, \overline{AP}(|A|^2,|B|^2),\, \overline{BQ}(|A|^2,|B|^2)),$$

where P *and* Q *are complex-valued polynomials in their arguments such that* $P(0,0) = Q(0,0) = 0$*.*

(ii) Assume that $\omega_1/\omega_2 = r/s \in \mathbb{Q}$*. Then the polynomial* $\mathbf{N}$ *in Theorem 5.2 is of the form*

$$\begin{aligned} \mathbf{N}(u) \;=\; & \Big(AP_1(|A|^2,|B|^2,A^s\overline{B}^r) + \overline{A}^{s-1}B^rP_2(|A|^2,|B|^2,\overline{A}^sB^r), \\ & BQ_1(|A|^2,|B|^2,\overline{A}^sB^r) + A^s\overline{B}^{r-1}Q_2(|A|^2,|B|^2,A^s\overline{B}^r), \\ & (\overline{AP_1(|A|^2,|B|^2,A^s\overline{B}^r)} + A^{s-1}\overline{B}^r\overline{P_2(|A|^2,|B|^2,\overline{A}^sB^r)}, \\ & \overline{BQ_1(|A|^2,|B|^2,\overline{A}^sB^r)} + \overline{A}^sB^{r-1}\overline{Q_2(|A|^2,|B|^2,A^s\overline{B}^r)}\Big), \end{aligned}$$

where P_1*,* P_2*,* Q_1*, and* Q_2 *are complex-valued polynomials in their arguments and* $P_1(0,0,0) = Q_1(0,0,0) = 0$*.*

Proof. We set

$$\mathbf{N}(u) = (\Phi_1(A,B,\overline{A},\overline{B}),\ \Phi_2(A,B,\overline{A},\overline{B}),\ \overline{\Phi}_1(A,B,\overline{A},\overline{B}),\ \overline{\Phi}_2(A,B,\overline{A},\overline{B})),$$

and then from (5.4) we find

$$\begin{aligned} \Phi_1(e^{-i\omega_1 t}A, e^{-i\omega_2 t}B, e^{i\omega_1 t}\overline{A}, e^{i\omega_2 t}\overline{B}) &= e^{-i\omega_1 t}\Phi_1(A,B,\overline{A},\overline{B}) \\ \Phi_2(e^{-i\omega_1 t}A, e^{-i\omega_2 t}B, e^{i\omega_1 t}\overline{A}, e^{i\omega_2 t}\overline{B}) &= e^{-i\omega_2 t}\Phi_2(A,B,\overline{A},\overline{B}) \end{aligned} \tag{5.24}$$

for all $t \in \mathbb{R}$, and $A, B \in \mathbb{C}$.

Consider the monomials

$$\phi^{(1)}_{p_1q_1p_2q_2} A^{p_1}\overline{A}^{q_1}B^{p_2}\overline{B}^{q_2} \quad \text{and} \quad \phi^{(2)}_{p_1q_1p_2q_2} A^{p_1}\overline{A}^{q_1}B^{p_2}\overline{B}^{q_2}$$

in the polynomials Φ_1 and Φ_2, respectively. Then (5.24) implies that

$$\omega_1(p_1 - q_1 - 1) + \omega_2(p_2 - q_2) = 0.$$

If $\omega_1/\omega_2 \notin \mathbb{Q}$, we then have

$$p_1 = q_1 + 1, \quad p_2 = q_2,$$

from which we conclude the result in part (i).

If $\omega_1/\omega_2 = r/s \in \mathbb{Q}$, then the relation above gives

$$r(p_1 - q_1 - 1) + s(p_2 - q_2) = 0,$$

and since r and s have no common divisor, we obtain

$$p_1 - q_1 - 1 = ls, \quad p_2 - q_2 = -lr$$

for some $l \in \mathbb{Z}$. For $l \geq 0$, this gives

$$p_1 = q_1 + 1 + ls, \quad q_2 = p_2 + lr,$$

which corresponds to a polynomial of the form $AP_1(|A|^2, |B|^2, A^s\overline{B}^r)$, where P_1 is a polynomial in its arguments. For $l = -l' < 0$, we find

$$q_1 = p_1 + s - 1 + (l' - 1)s, \quad p_2 = q_2 + r + (l' - 1)r,$$

which gives a polynomial of the form $\overline{A}^{s-1}B^r P_2(|A|^2, |B|^2, \overline{A}^s B^r)$, where P_2 is a polynomial in its arguments. The same arguments work for the polynomial Φ_2. Notice that the lowest order terms in these polynomials, which are not of the standard form found in the irrational case, are of degree $r + s - 1 \geq 2$ (we assumed $\omega_1 \neq \omega_2$, which implies that r and s are different positive integers). This ends the proof of the lemma. ∎

Exercise 5.16 (Generalization)**.** *Consider the matrix* $\mathbf{L}$ *in* $\mathbb{R}^{2n}$ *with the pairs of simple eigenvalues* $\pm i\omega_1, \ldots, \pm i\omega_n$.

(i) Assume that $\langle \alpha, \omega \rangle \neq 0$ for any $\alpha \in \mathbb{Z}^n \setminus \{0\}$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$, and $\omega = (\omega_1, \ldots, \omega_n)$. Show that the polynomial $\mathbf{N}$ in Theorem 5.2 is of form

$$\begin{aligned}\mathbf{N}(u) &= (A_1P_1(|A_1|^2, \ldots, |A_n|^2), \ldots, A_nP_n(|A_1|^2, \ldots, |A_n|^2), \\ &\quad \overline{A_1P_1}(|A_1|^2, \ldots, |A_n|^2), \ldots, \overline{A_nP_n}(|A_1|^2, \ldots, |A_n|^2)),\end{aligned}$$

where the P_j, $j = 1, \ldots, n$, are complex-valued polynomials in their arguments such that $P_j(0, \ldots, 0) = 0$.

(ii) Set $|\alpha_0| = \min\{|\alpha| ; \langle \alpha, \omega \rangle = 0, \alpha \in \mathbb{Z}^n \setminus \{0\}\} < \infty$, where $|\alpha| = \sum_{j=1}^n |\alpha_j|$, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. Show that the lowest order terms in the polynomial $\mathbf{N}$ in Theorem 5.2, which are not of the "standard form" obtained in the case (i), are of degree $|\alpha_0|-1$.

In the remainder of this section, we give the normal forms in the cases $(i\omega)^2$, $0^2(i\omega)$. The proofs of the following results are given in Appendices C2, C3 and C4 of [21]. The proofs can be also found in [14]. We also refer to [28], [12] for different proofs of the results in the cases $(i\omega)^2$ and $0^2 0^2$.

Lemma 5.17 ($(i\omega)^2$ normal form)**.** *Assume that the matrix $\mathbf{L}$ is of the form*

$$\mathbf{L} = \begin{pmatrix} i\omega & 1 & 0 & 0 \\ 0 & i\omega & 0 & 0 \\ 0 & 0 & -i\omega & 1 \\ 0 & 0 & 0 & -i\omega \end{pmatrix},$$

where $\omega > 0$, in a basis of $\mathbb{R}^4$ in which a vector $u \in \mathbb{R}^4$ is represented by $u = (A, B, \overline{A}, \overline{B})$, with $A, B \in \mathbb{C}$. Then the polynomial $\mathbf{N}$ in Theorem 5.2 is of the form

$$\begin{aligned}\mathbf{N}(u) &= (AP(|A|^2, i(A\overline{B} - \overline{A}B)),\ BP(|A|^2, i(A\overline{B} - \overline{A}B)) + AQ(|A|^2, i(A\overline{B} - \overline{A}B)), \\ &\quad \overline{AP}(|A|^2, i(A\overline{B} - \overline{A}B)),\ \overline{BP}(|A|^2, i(A\overline{B} - \overline{A}B)) + \overline{AQ}(|A|^2, i(A\overline{B} - \overline{A}B))),\end{aligned}$$

where P and Q are complex-valued polynomials in their arguments, satisfying $P(0,0) = Q(0,0) = 0$.

Lemma 5.18 (0^2(iω) normal form)**.** *Assume that the matrix $\mathbf{L}$ is of the form*

$$\mathbf{L} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i\omega & 0 \\ 0 & 0 & 0 & -i\omega \end{pmatrix},$$

where $\omega > 0$, in a basis of $\mathbb{R}^4$ in which a vector $u \in \mathbb{R}^4$ is represented by $u = (A, B, C, \overline{C})$, with $A, B \in \mathbb{R}$ and $C \in \mathbb{C}$. Then the polynomial $\mathbf{N}$ in Theorem 5.2 is of the form

$$\begin{aligned}N(u) &= (AP_0(A, |C|^2),\ BP_0(A, |C|^2) + P_1(A, |C|^2), \\ &\quad CP_2(A, |C|^2),\ \overline{CP_2}(A, |C|^2)),\end{aligned}$$

where P_0 and P_1 are real-valued polynomials, and P_2 is a complex-valued polynomial, satisfying

$$P_0(0,0) = P_1(0,0) = P_2(0,0) = \frac{\partial P_1}{\partial A}(0,0) = 0.$$

Remark 5.19. *The interested reader may find other normal forms in literature, as for example* $0^2 0^3$ *in [14],* $(i\omega_1)^2(i\omega_2)$ *in [29, 50],* $(i\omega)^5$ *with spherical symmetry* $O(3)$ *in [33].*

5.2 Parameter-Dependent Normal Forms

5.2.1 Main Result

In the same framework as above, we are interested now in parameter-dependent equations of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u, \mu), \tag{5.25}$$

in which we assume that $\mathbf{L}$ and $\mathbf{R}$ satisfy the following hypothesis.

Hypothesis 5.20. *Assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in (5.25) have the following properties:*

(i) $\mathbf{L}$ *is a linear map in* $\mathbb{R}^n$*;*

(ii) for some $k \geq 2$*, there exist neighborhoods* $\mathcal{V}_u \subset \mathbb{R}^n$ *and* $\mathcal{V}_\mu \subset \mathbb{R}^m$ *of* 0 *such that* $\mathbf{R} \in \mathcal{C}^k(\mathcal{V}_u \times \mathcal{V}_\mu, \mathbb{R}^n)$ *and*

$$\mathbf{R}(0,0) = 0, \quad D_u\mathbf{R}(0,0) = 0.$$

In this situation we have the following result.

Theorem 5.21 (Normal form for perturbed vector fields)**.** *Assume that Hypothesis 5.20 holds. Then for any positive integer p,* $2 \leq p \leq k$*, there exist neighborhoods* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *of* 0 *in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively, such that for any* $\mu \in \mathcal{V}_2$*, there is a polynomial* $\mathbf{\Phi}_\mu : \mathbb{R}^n \to \mathbb{R}^n$ *of degree p with the following properties:*

(i) The coefficients of the monomials of degree q in $\mathbf{\Phi}_\mu$ *are functions of* μ *of class* $\mathcal{C}^{k-q}$*, and*

$$\mathbf{\Phi}_0(0) = 0, \quad D_u\mathbf{\Phi}_0(0) = 0.$$

(ii) For $v \in \mathcal{V}_1$*, the polynomial change of variable*

$$u = v + \mathbf{\Phi}_\mu(v), \tag{5.26}$$

transforms equation (5.25) into the "normal form"

$$\frac{dv}{dt} = \mathbf{L}v + \mathbf{N}_\mu(v) + \boldsymbol{\rho}(v, \mu), \tag{5.27}$$

and the following properties hold:

(a) For any $\mu \in \mathcal{V}_2$*,* $\mathbf{N}_\mu$ *is a polynomial* $\mathbb{R}^n \to \mathbb{R}^n$ *of degree p, with coefficients depending upon* μ*, such that the coefficients of the monomials of degree q are of class* $\mathcal{C}^{k-q}$*, and*

$$\mathbf{N}_0(0) = 0, \quad D_v\mathbf{N}_0(0) = 0.$$

(b) The equality

$$\mathbf{N}_\mu(e^{t\mathbf{L}^*}v) = e^{t\mathbf{L}^*}\mathbf{N}_\mu(v) \tag{5.28}$$

holds for all $(t, v) \in \mathbb{R} \times \mathbb{R}^n$ *and* $\mu \in \mathcal{V}_2$.

(c) The map $\boldsymbol{\rho}$ *belongs to* $\mathcal{C}^k(\mathcal{V}_1 \times \mathcal{V}_2, \mathbb{R}^n)$*, and*

$$\boldsymbol{\rho}(v, \mu) = o(\|v\|^p)$$

for all $\mu \in \mathcal{V}_2$.

The proof of this theorem is given in Appendix C5 of [21]. We point out that in most results on normal forms in the literature the normal form $\mathbf{N}_\mu$ is a polynomial in both v and μ, whereas here it is only a polynomial in v which is better for some use.

Remark 5.22. *(i) As for Theorem 5.2, identity (5.28) is equivalent to the identity*

$$D_v\mathbf{N}_\mu(v)\mathbf{L}^*v = \mathbf{L}^*\mathbf{N}_\mu(v) \text{ for all } v \in \mathbb{R}^n,\ \mu \in \mathcal{V}_2.$$

(ii) Notice that the origin is not necessarily an equilibrium of (5.25) when $\mu \neq 0$*. Then* $\mathbf{N}_\mu(0)$ *is, in general, not* 0*, and the equality above shows that in this case*

$$\mathbf{N}_\mu(0) \in \ker \mathbf{L}^*.$$

(iii) In Theorem 5.21, the polynomials $\mathbf{\Phi}_\mu$ *and* $\mathbf{N}_\mu$ *have coefficients depending upon* μ*. The regularity with respect to* μ *of these coefficients decreases as the degree of the corresponding monomial increases. In applications, we actually compute the Taylor expansions of the coefficients of the polynomials* $\mathbf{\Phi}_\mu$ *and* $\mathbf{N}_\mu$ *up to a needed degree in* μ *(see Section 5.2.3 below). Also notice, that the remainder* $\boldsymbol{\rho}$ *in (5.27) is uniformly estimated for* $\mu \in \mathcal{V}_2$*. This is sometimes useful when one is looking for the optimal behavior of certain solutions as* $t \to \pm\infty$*.*

(iv) We can consider again the examples in Sections 5.1.2–5.1.4, now in the context of the parameter-dependent equation (5.25). In each case, we find that the parameter-dependent normal form polynomial $\mathbf{N}_\mu$ *has the same structure as the unperturbed polynomial* $\mathbf{N}$*, but now with coefficients depending upon the parameter* μ*.*

5.2.2 Linear Normal Forms

An interesting particular case occurs when the map $\mathbf{R}(u, \mu)$ is linear in u, so that we have a linear equation

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}_\mu u.$$

Assuming that $\mathbf{R}_0 = 0$, Hypothesis 5.20 is satisfied and the result in Theorem 5.21 holds. The polynomial $\mathbf{\Phi}_\mu$ is of degree 1 in this case, so that we have a linear change of variables. The normal form is also linear,

$$\frac{dv}{dt} = (\mathbf{L} + \mathbf{N}_\mu)v,$$

in which the map $\mu \mapsto \mathbf{N}_\mu$ is of class $\mathcal{C}^{k-1}$ in a neighborhood of 0, and now

$$\mathbf{N}_\mu \mathbf{L}^* = \mathbf{L}^* \mathbf{N}_\mu. \tag{5.29}$$

This result was proved in [2] and is of particular interest since it gives a *smooth* unfolding of a linear map $\mathbf{L}$, which is, in general, not the case when one uses the classical transformation into Jordan form. For example, assume that $\mathbf{L}$ is not diagonalizable, but $\mathbf{L} + \mathbf{R}_\mu$ is diagonalizable for $\mu \neq 0$. Then the linear change of variables, which transforms $\mathbf{L} + \mathbf{R}_\mu$ into a diagonal matrix, is singular in $\mu = 0$.

Exercise 5.23. *Consider the* 3×3*-matrix*

$$\mathbf{L} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

in which λ *is a real parameter, and consider a linear perturbation* $\mathbf{R}_\mu$ *depending smoothly upon* $\mu \in \mathbb{R}^m$*, such that* $\mathbf{R}_0 = 0$.

(i) Assume that $\lambda \neq 0$*. Show that there is a linear change of variables in* $\mathbb{R}^3$*, which is smooth in* μ *in a neighborhood of 0, such that the transformed matrix is of the form*

$$\begin{pmatrix} \alpha_\mu & 1 & 0 \\ \beta_\mu & \alpha_\mu & 0 \\ 0 & 0 & \lambda + \gamma_\mu \end{pmatrix},$$

where α_μ*,* β_μ*, and* γ_μ *depend smoothly upon* μ*. Compute the first two leading order terms in the Taylor expansions in* μ *of the vectors in the basis* $\{\zeta_1(\mu), \zeta_2(\mu), \zeta_3(\mu)\}$ *of* $\mathbb{R}^3$ *consisting of generalized eigenvectors of the new matrix, which is the smooth continuation of the basis* $\{\xi_1, \xi_2, \xi_3\}$ *such that*

$$\mathbf{L}\xi_1 = 0, \quad \mathbf{L}\xi_2 = \xi_1, \quad \mathbf{L}\xi_3 = 0.$$

Hint: *Use (5.29) to prove the first part. For the second part, use the dual basis* $\{\xi_1^*, \xi_2^*, \xi_3^*\}$ *such that*

$$\mathbf{L}^*\xi_1^* = \xi_2^*, \quad \mathbf{L}^*\xi_2^* = 0, \quad \mathbf{L}^*\xi_3^* = \lambda\xi_3^*, \quad \langle \xi_j, \xi_l^* \rangle = \delta_{kl},$$

and identify the different powers of μ *in the identities*

$$\begin{aligned} (\mathbf{L} + \mathbf{R}_\mu)\zeta_1(\mu) &= \alpha_\mu \zeta_1(\mu) + \beta_\mu \zeta_2(\mu), \\ (\mathbf{L} + \mathbf{R}_\mu)\zeta_2(\mu) &= \zeta_1(\mu) + \alpha_\mu \zeta_2(\mu), \\ (\mathbf{L} + \mathbf{R}_\mu)\zeta_3(\mu) &= (\lambda + \gamma(\mu))\zeta_3(\mu). \end{aligned}$$

(ii) Assume that $\lambda = 0$. Show that there is a linear change of variables in $\mathbb{R}^3$, which is smooth in μ in a neighborhood of 0, such that the transformed matrix is of the form

$$\begin{pmatrix} \alpha_\mu & 1 & 0 \\ \beta_\mu & \alpha_\mu & \varepsilon_\mu \\ \delta_\mu & 0 & \gamma_\mu \end{pmatrix},$$

where α_μ, β_μ, γ_μ, δ_μ, and ε_μ depend smoothly upon μ. Describe a method for computing the Taylor expansions in μ of the vectors in the basis $\{\zeta_1(\mu), \zeta_2(\mu), \zeta_3(\mu)\}$ of $\mathbb{R}^3$ consisting of generalized eigenvectors of the new matrix, which is the smooth continuation of the basis $\{\xi_1, \xi_2, \xi_3\}$ above. Show that in general the eigenvalues of the transformed matrix do not depend smoothly upon μ, even for a single parameter $\mu \in \mathbb{R}$.

5.2.3 Derivation of the Parameter-Dependent Normal Form

In this section we give a method of computing the Taylor expansions of the polynomials $\mathbf{\Phi}_\mu$ and $\mathbf{N}_\mu$ given by Theorem 5.21. We have already used this method in the particular case of the Hopf bifurcation in Section 3.1, and without parameters in the example of a 0^2 normal form in Section 5.1.2.

We write the Taylor expansion of $\mathbf{R}$ and rewrite polynomials $\mathbf{\Phi}_\mu$ and $\mathbf{N}_\mu$ as follows:

$$\begin{aligned}
\mathbf{R}(u,\mu) &= \sum_{1\leq q+l\leq p} \mathbf{R}_{ql}(u^{(q)},\mu^{(l)}) + o((\|u\|+\|\mu\|)^p), \quad \mathbf{R}_{10}=0,\\
\mathbf{\Phi}_\mu(v) &= \sum_{1\leq q+l\leq p} \mathbf{\Phi}_{ql}(v^{(q)},\mu^{(l)}) + o((\|v\|+\|\mu\|)^p), \quad \mathbf{\Phi}_{10}=0,\\
\mathbf{N}_\mu(v) &= \sum_{1\leq q+l\leq p} \mathbf{N}_{ql}(v^{(q)},\mu^{(l)}) + o((\|v\|+\|\mu\|)^p), \quad \mathbf{N}_{10}=0,
\end{aligned}$$

where $\mathbf{R}_{ql}$, $\mathbf{\Phi}_{ql}$, and $\mathbf{N}_{ql}$ are $(q+l)$-linear maps on $(\mathbb{R}^n)^q \times (\mathbb{R}^m)^l$, $u^{(q)} = (u,\dots,u) \in (\mathbb{R}^n)^q$, and $\mu^{(l)} = (\mu,\dots,\mu) \in (\mathbb{R}^m)^l$. Furthermore, $\mathbf{R}_{ql}(\cdot,\mu^{(l)})$ and $\mathbf{R}_{ql}(u^{(q)},\cdot)$ are q-linear symmetric and l-linear symmetric, respectively, and similar properties hold for $\mathbf{\Phi}_{ql}$, and $\mathbf{N}_{ql}$. Notice that the terms $o((\|v\|+\|\mu\|)^p)$ in the expansions of $\mathbf{\Phi}$ and $\mathbf{N}$ come from the fact that these are polynomials in v with coefficients that are functions of μ, of class $\mathcal{C}^{k-q}$ for the monomials of degree q.

Now we proceed as in the proof of Theorem 5.2. Differentiating (5.26) with respect to t and replacing du/dt and dv/dt from (5.25) and (5.27), respectively, we obtain the identity

$$\mathcal{A}_{\mathbf{L}}\mathbf{\Phi}_\mu(v) + \mathbf{N}_\mu(v) = \mathbf{\Pi}_p\left(\mathbf{R}(v+\mathbf{\Phi}_\mu(v),\mu) - D_v\mathbf{\Phi}_\mu(v)\mathbf{N}_\mu(v)\right). \tag{5.30}$$

Here $\mathcal{A}_{\mathbf{L}}$ is the homological operator given by (5.10), and $\mathbf{\Pi}_p$ represents the linear map which associates to a map of class $\mathcal{C}^p$ the polynomial of degree p in its Taylor expansion. Identifying the coefficients of the monomials of degree q in u and of degree 0 in μ leads to

$$\begin{aligned}
\mathcal{A}_{\mathbf{L}}\mathbf{\Phi}_{20} + \mathbf{N}_{20} &= \mathbf{R}_{20},\\
\mathcal{A}_{\mathbf{L}}\mathbf{\Phi}_{30} + \mathbf{N}_{30} &= \mathbf{Q}_{30},
\end{aligned}$$

with

$$\mathbf{Q}_{30}(v^{(3)}) = \mathbf{R}_{30}(v^{(3)}) + 2\mathbf{R}_{20}(v,\mathbf{\Phi}_{20}(v)) - 2\mathbf{\Phi}_{20}(v,\mathbf{N}_{20}(v^{(2)}))$$

for $q = 2$ and $q = 3$, respectively, and similar equalities hold for $q \geq 4$, just as in (5.8). Then the equation for $\mathcal{A}_{\mathbf{L}}\mathbf{\Phi}_{q0} + \mathbf{N}_{q0}$ only contains in the right hand side terms involving $\mathbf{\Phi}_{q'0}$ and $\mathbf{N}_{q'0}$, with $q' \leq q-1$, so that we can successively determine $\mathbf{\Phi}_{q0}$ and $\mathbf{N}_{q0}$.

Next, we consider the monomials of degree q in u and of degree 1 in μ, and obtain

$$\begin{aligned}\mathcal{A}_{\mathbf{L}}\mathbf{\Phi}_{01} + \mathbf{N}_{01} &= \mathbf{R}_{01},\\ \mathcal{A}_{\mathbf{L}}\mathbf{\Phi}_{11}(v,\mu) + \mathbf{N}_{11}(v,\mu) &= \mathbf{R}_{11}(v,\mu) - 2\mathbf{\Phi}_{20}(v,\mathbf{N}_{01}(\mu))\end{aligned}$$

for $q = 0$ and $q = 2$, respectively, and

$$\mathcal{A}_{\mathbf{L}}\mathbf{\Phi}_{q1}(v^{(q)},\mu) + \mathbf{N}_{q1}(v^{(q)},\mu) = \mathbf{Q}_{q1}(v^{(q)},\mu)$$

for $q \geq 2$, where $\mathbf{Q}_{q1}$ depends upon $\mathbf{\Phi}_{q'1}, \mathbf{N}_{q'1}, \mathbf{\Phi}_{q''0}, \mathbf{N}_{q''0}$ such that $q' \leq q-1$ and $q'' \leq q+1$. Consequently, once we have found $(\mathbf{\Phi}_{q0}, \mathbf{N}_{q0})$, $q = 2, \dots, p$, we can determine $(\mathbf{\Phi}_{q1}, \mathbf{N}_{q1})$ by successively solving the equations above for $q = 0, 1, \dots, p-1$. More generally, we obtain

$$\mathcal{A}_{\mathbf{L}}\mathbf{\Phi}_{ql}(v^{(q)},\mu^{(l)}) + \mathbf{N}_{ql}(v^{(q)},\mu^{(l)}) = \mathbf{Q}_{ql}(v^{(q)},\mu^{(l)}),$$

which is of the same form as above, with $\mathbf{Q}_{ql}$ depending upon $\mathbf{\Phi}_{q'l'}$ and $\mathbf{N}_{q'l'}$ either such that $q'+l' \leq q+l-1$ with $l' \leq l$, or such that $q'+l' = q+l$ with $l' \leq l-1$. This shows that once we found $(\mathbf{\Phi}_{qj}, \mathbf{N}_{qj})$, for $q+j \leq p$, $j = 0, 1, \dots, l$, then we can determine $(\mathbf{\Phi}_{q'l'}, \mathbf{N}_{q'l'})$ for $l' = l+1$ and $q' \leq p-l-1$. We indicate in Figure 5.1 the way in which $(\mathbf{\Phi}_{ql}, \mathbf{N}_{ql})$ depend upon $(\mathbf{\Phi}_{q'l'}, \mathbf{N}_{q'l'})$.

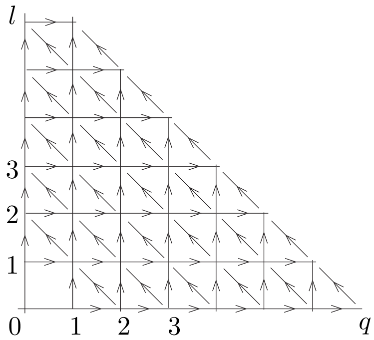


Figure 5.1: Plot of the indices (q,l) of $(\mathbf{\Phi}_{ql}, \mathbf{N}_{ql})$. The arrows indicate the dependence of $(\mathbf{\Phi}_{ql}, \mathbf{N}_{ql})$ at the position (q,l) upon $(\mathbf{\Phi}_{q'l'}, \mathbf{N}_{q'l'})$ at the position (q',l').

5.2.4 Example: 0^2 Normal Form with Parameters

Consider the second order differential equation

$$u'' = \mu_0 + \mu_1 u + \mu_2 u' + \alpha u^2 + \beta u u' + \gamma (u')^2,$$

where α, β, γ are real constants, and μ_0, μ_1, μ_2 small parameters. Notice that for $\mu_0 = \mu_1 = \mu_2 = 0$ this is precisely the equation (5.17) for which the normal form has been computed in Section 5.1.2. Therefore, it remains to compute the terms in the normal form involving the three small parameters μ_0, μ_1, and μ_2.

Normal Form We set $U = (u, v)$ and $\mu = (\mu_0, \mu_1, \mu_2) \in \mathbb{R}^3$, so that the equation becomes

$$\frac{dU}{dt} = \mathbf{L}U + \mathbf{R}(U, \mu), \qquad \mathbf{R}(U, \mu) = \mathbf{R}_{01}(\mu) + \mathbf{R}_2(U, U) + \mathbf{R}_{11}(U, \mu), \tag{5.31}$$

where $\mathbf{L}$ and $\mathbf{R}_2$ are as in (5.18), and

$$\mathbf{R}_{01}(\mu) = \begin{pmatrix} 0 \\ \mu_0 \end{pmatrix}, \quad \mathbf{R}_{11}(U, \mu) = \begin{pmatrix} 0 \\ \mu_1 u + \mu_2 v \end{pmatrix}.$$

We are interested in computing the normal form of this system up to terms of order 2, so that it is enough to consider a polynomial $\mathbf{\Phi}_\mu$ of degree 2,

$$\mathbf{\Phi}_\mu(A, B) = \mathbf{\Phi}_{001}(\mu) + A\mathbf{\Phi}_{101}(\mu) + B\mathbf{\Phi}_{011}(\mu) + A^2\mathbf{\Phi}_{200} + AB\mathbf{\Phi}_{110} + B^2\mathbf{\Phi}_{020},$$

where $\mathbf{\Phi}_{ij1} : \mathbb{R}^3 \to \mathbb{R}^2$ are linear maps. Since for $\mu = 0$ the result is the same as the one found for the equation (5.17) in Section 5.1.2, it is clear that here

$$\mathbf{\Phi}_{200} = \mathbf{\Phi}_{20}, \quad \mathbf{\Phi}_{110} = \mathbf{\Phi}_{11}, \quad \mathbf{\Phi}_{020} = \mathbf{\Phi}_{02},$$

where

$$\mathbf{\Phi}_{20} = \frac{\gamma}{2}\zeta_0, \quad \mathbf{\Phi}_{11} = \gamma\zeta_1, \quad \mathbf{\Phi}_{02} = 0$$

have been computed in Section 5.1.2. According to Lemma 5.9 and Remark 5.10(ii), and taking into account the result found for $\mu = 0$ in Section 5.1.2, it follows that the change of variables

$$U = A\zeta_0 + B\zeta_1 + \mathbf{\Phi}_\mu(A, B), \tag{5.32}$$

where

$$\zeta_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

transforms the system (5.31) into the normal form

$$\begin{aligned} \frac{dA}{dt} &= B \\ \frac{dB}{dt} &= \alpha_1(\mu) + \alpha_2(\mu)A + \alpha_3(\mu)B + \alpha A^2 + \beta AB \\ &\qquad + O(|\mu|^2 + |\mu|(|A| + |B|)^2 + (|A| + |B|)^3), \end{aligned} \tag{5.33}$$

in which $\alpha_j : \mathbb{R}^3 \to \mathbb{R}$, $j = 0, 1, 2$, are linear maps.

Computation of α_0***,*** α_1***, and*** α_2 We proceed as indicated in Section 5.2.3, and also as in the previous computations. We substitute the change of variables (5.32) into the system (5.31), and then replace the derivatives dA/dt and dB/dt from (5.33). In the resulting equality we now identify the terms of orders $O(\mu)$, $O(\mu A)$, and $O(\mu B)$, which gives the equations

$$\begin{aligned}
\alpha_1(\mu)\zeta_1 &= \mathbf{L}\mathbf{\Phi}_{001}(\mu) + \mathbf{R}_{01}(\mu), &(5.34)\\
\alpha_2(\mu)\zeta_1 + \alpha_1(\mu)\mathbf{\Phi}_{110} &= \mathbf{L}\mathbf{\Phi}_{101}(\mu) + \mathbf{R}_{11}(\zeta_0,\mu)\\
&\quad + 2\mathbf{R}_2(\zeta_0, \mathbf{\Phi}_{001}(\mu)), &(5.35)\\
\alpha_3(\mu)\zeta_1 + 2\alpha_1(\mu)\mathbf{\Phi}_{020} + \mathbf{\Phi}_{101}(\mu) &= \mathbf{L}\mathbf{\Phi}_{011}(\mu) + \mathbf{R}_{11}(\zeta_1,\mu)\\
&\quad + 2\mathbf{R}_2(\zeta_1, \mathbf{\Phi}_{001}(\mu)). &(5.36)
\end{aligned}$$

Using the fact that the range $im\mathbf{L}$ of $\mathbf{L}$ is given by $im\mathbf{L} = \{(u,0); u \in \mathbb{R}\} \subset \mathbb{R}^2$ and that the kernel $\ker \mathbf{L}$ is spanned by ζ_0, we can solve these equations and determine α_j from the corresponding solvability conditions.

Solving these three equations we find, successively,

$$\alpha_1(\mu) = \mu_0, \quad \mathbf{\Phi}_{001}(\mu) = \phi_{001}(\mu)\zeta_0,$$

$$\alpha_2(\mu) = -\gamma\mu_0 + \mu_1 + 2\alpha\phi_{001}(\mu), \quad \mathbf{\Phi}_{101}(\mu) = \phi_{101}(\mu)\zeta_0,$$

and

$$\alpha_3(\mu) = \mu_2 + \beta\phi_{001}(\mu), \quad \mathbf{\Phi}_{011}(\mu) = \phi_{101}(\mu)\zeta_1 + \phi_{011}(\mu)\zeta_0,$$

in which $\phi_{001}, \phi_{101}, \phi_{011} : \mathbb{R}^3 \to \mathbb{R}$ are arbitrary linear maps. A simple choice is of course $\phi_{001} = \phi_{101} = \phi_{011} = 0$, which then gives

$$\alpha_1(\mu) = \mu_0, \quad \alpha_2(\mu) = -\gamma\mu_0 + \mu_1, \quad \alpha_3(\mu) = \mu_2.$$

Alternatively, if $\beta \neq 0$ we may choose $\phi_{001}(\mu)$ such that $\alpha_3(\mu) = 0$, i.e.,

$$\phi_{001}(\mu) = -\frac{\mu_2}{\beta},$$

which gives

$$\alpha_1(\mu) = \mu_0, \quad \alpha_2(\mu) = -\gamma\mu_0 + \mu_1 - \frac{2\alpha}{\beta}\mu_2, \quad \alpha_3(\mu) = 0,$$

whereas if $\alpha \neq 0$ we may choose $\phi_{001}(\mu)$ such that $\alpha_2(\mu) = 0$, i.e.,

$$\phi_{001}(\mu) = \frac{1}{2\alpha}\left(\gamma\mu_0 - \mu_1\right),$$

which gives

$$\alpha_1(\mu) = \mu_0, \quad \alpha_2(\mu) = 0, \quad \alpha_3(\mu) = \mu_2 + \frac{\beta}{2\alpha}\left(\gamma\mu_0 - \mu_1\right).$$

Actually, these choices can be made in general for a Takens–Bogdanov bifurcation (see Section 5.4.5).

5.3 Symmetries and Reversibility

In this section, we consider the particular cases where the equation is equivariant under the action of a symmetry and where it possesses a reversibility symmetry. In both cases we show that the symmetry is inherited by the normal form. We state our results for equation (5.1), but the same results also hold for the parameter-dependent equation (5.25).

5.3.1 Equivariant Vector Fields

We start with the case of an equation that is equivariant under the action of a linear symmetry. More precisely, we make the following assumption.

Hypothesis 5.24 (Equivariant vector field)**.** *Assume that there exists an isometry* $\mathbf{T} \in \mathcal{L}(\mathbb{R}^n)$ *which commutes with the vector field in the equation (5.1),*

$$\mathbf{TL}u = \mathbf{LT}u, \quad \mathbf{TR}(u) = \mathbf{R}(\mathbf{T}u) \text{ for all } u \in \mathbb{R}^n.$$

In this situation, the following result holds:

Theorem 5.25 (Equivariant normal forms)**.** *Under the assumptions of Theorem 5.2, further assume that Hypothesis 5.24 holds. Then the polynomials* $\mathbf{\Phi}$ *and* $\mathbf{N}$ *in Theorem 5.2 commute with* $\mathbf{T}$*.*

5.3.2 Reversible Vector Fields

Next, we consider the case of reversible equations for which we assume that the following assumptions are satisfied.

Hypothesis 5.26 (Reversible vector field)**.** *Assume that there exists an isometry* $\mathbf{S} \in \mathcal{L}(\mathbb{R}^n)$*, with*

$$\mathbf{S}^2 = \mathbb{I}, \quad \mathbf{S} \neq \mathbb{I},$$

and which anticommutes with the vector field in (5.1),

$$\mathbf{SL}u = -\mathbf{LS}u, \quad \mathbf{SR}(u) = -\mathbf{R}(\mathbf{S}u) \text{ for all } u \in \mathbb{R}^n.$$

In this case the following result holds:

Theorem 5.27 (Reversible normal forms)**.** *Under the assumptions of Theorem 5.2, further assume that Hypothesis (5.26) holds. Then the polynomial* $\mathbf{\Phi}$ *in Theorem 5.2 commutes with* $\mathbf{S}$*, whereas the polynomial* $\mathbf{N}$ *anticommutes with* $\mathbf{S}$*.*

5.3.3 Example: van der Pol System

Consider the van der Pol system [59],

$$\begin{aligned} u_1' &= \mu u_1 - u_2 - u_1^3 \\ u_2' &= u_1, \end{aligned}$$

in which μ is a small parameter. (This system models an electrical circuit with a triode vacuum tube, nowadays replaced by a transistor.) Notice that the system is invariant under the reflection $(u_1, u_2) \mapsto -(u_1, u_2)$.

Normal Form We set $U = (u_1, u_2)$, so that the system is of the form

$$\frac{dU}{dt} = \mathbf{L}U + \mathbf{R}(U, \mu), \qquad \mathbf{R}(U, \mu) = \mu\mathbf{R}_{11}(U) + \mathbf{R}_{30}(U, U, U), \tag{5.37}$$

where

$$\mathbf{L} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{R}_{11}(U) = \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \quad \mathbf{R}_{30}(U, V, W) = \begin{pmatrix} -u_1 v_1 w_1 \\ 0 \end{pmatrix}.$$

Due to the reflection invariance mentioned above, the system (5.37) is equivariant under the action of

$$\mathbf{T} = -\mathbb{I}.$$

The linear map $\mathbf{L}$ has a pair of complex conjugated eigenvalues $\pm i$, with associated eigenvectors

$$\zeta = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \overline{\zeta} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

This implies that $\mu = 0$ is a bifurcation point, at which we expect a Hopf bifurcation to occur. We are interested in computing the normal form of this system up to terms of order 3, taking into account the equivariance of the system under the action of $\mathbf{T}$.

We consider the change of variables

$$U = A\zeta + \overline{A\zeta} + \mathbf{\Phi}_\mu(A, \overline{A}),$$

with $A(t) \in \mathbb{C}$ and $\mathbf{\Phi}_\mu$ a polynomial of degree 3, since we are interested in the normal form up to terms of order 3. According to the result in Lemma 5.7 and Theorem 5.25 there exists a polynomial $\mathbf{\Phi}_\mu$ which commutes with $\mathbf{T}$ and such that the system is transformed into the normal form

$$\frac{dA}{dt} = iA + a\mu A + bA|A|^2 + O(\mu^2|A| + |\mu||A|^3| + |A|^5).$$

Since $\mathbf{\Phi}_\mu$ commutes with $\mathbf{T}$, it follows that $\mathbf{\Phi}_\mu$ is an odd polynomial, hence

$$\mathbf{\Phi}_\mu(A, \overline{A}) = \mu A\mathbf{\Phi}_{101} + \mu\overline{A}\mathbf{\Phi}_{011} + A^3\mathbf{\Phi}_{300} + A^2\overline{A}\mathbf{\Phi}_{210} + A\overline{A}^2\mathbf{\Phi}_{120} + \overline{A}^3\mathbf{\Phi}_{030}.$$

Computation of the Coefficients a ***and*** b We proceed as in the computation of the Hopf bifurcation in Section 3.1, which leads for a general Hopf bifurcation to the system (3.22)–(3.27). Here, due to the equivariance under $\mathbf{T}$, implying in particular that $\mathbf{\Phi}_\mu$ is an odd polynomial, several terms in this calculation vanish, so that we find the system

$$a\zeta + (i - \mathbf{L})\mathbf{\Phi}_{101} = \mathbf{R}_{11}(\zeta) \tag{5.38}$$

$$(3i - \mathbf{L})\mathbf{\Phi}_{300} = \mathbf{R}_{30}(\zeta, \zeta, \zeta) \tag{5.39}$$

$$b\zeta + (i - \mathbf{L})\mathbf{\Phi}_{210} = 3\mathbf{R}_{30}(\zeta, \zeta, \overline{\zeta}), \tag{5.40}$$

instead of the general system (3.22)–(3.27). Now, the coefficients a and b are easily computed from the solvability conditions for the equations (5.38) and (5.40). Recall that these conditions are orthogonality conditions on the kernel of the adjoint matrix, namely,

$$(i - \mathbf{L})^* = -i - \mathbf{L}^* = -i + \mathbf{L},$$

e.g., see Section 3.1, which is here one-dimensional and spanned by

$$\zeta^* = \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

This vector is chosen such that $\langle \zeta, \zeta_* \rangle = 1$, and then the solvability conditions lead to

$$a = \langle \mathbf{R}_{11}(\zeta), \zeta^* \rangle = \frac{1}{2}, \quad b = \langle 3\mathbf{R}_{30}(\zeta, \zeta, \overline{\zeta}), \zeta^* \rangle = -\frac{3}{2}.$$

Notice that $b < 0$, which implies that we have a *supercritical Hopf bifurcation.* Since $a > 0$ the branch of stable periodic solutions bifurcates for $\mu > 0$.

Exercise 5.28. *Compute the higher orders terms and show that*

$$\begin{aligned} \mathbf{\Phi}_\mu(A, \overline{A}) &= \mu A \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} + A^3 \begin{pmatrix} 3i/8 \\ 1/8 \end{pmatrix} + A^2\overline{A} \begin{pmatrix} 0 \\ -3/2 \end{pmatrix} \\ &\quad + \overline{A}^3 \begin{pmatrix} -3i/8 \\ 1/8 \end{pmatrix} + A\overline{A}^2 \begin{pmatrix} 0 \\ -3/2 \end{pmatrix} + O(|A|^5 + |\mu||A|^3 + |\mu|^2|A|), \end{aligned}$$

and that the normal form is

$$\frac{dA}{dt} = \left(i + \frac{1}{2}\mu - \frac{i}{8}\mu^2 \right) A - \frac{3}{2}(1 - i\mu) A|A|^2 - \frac{63i}{16} A|A|^4 + h.o.t..$$

5.4 Normal Forms for Reduced Systems on Center Manifolds

Consider an infinite-dimensional system of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u, \mu), \tag{5.41}$$

which satisfies the assumptions in center manifold Theorem 4.18. Then the reduced system is of the form (5.25) and satisfies Hypothesis 5.20, so that we can apply normal form Theorem 5.21. We show in this section how to compute the normal form of the reduced system directly from the infinite-dimensional, *without computing the reduced system.* Of course, this is the most efficient way of computation in applications.

5.4.1 Computation of Center Manifolds and Normal Forms

Recall that the center manifold theorem gives solutions of the form

$$u = u_0 + \mathbf{\Psi}(u_0, \mu),$$

with $u_0 \in \mathcal{E}_0$ and $\mathbf{\Psi}(u_0, \mu) \in \mathcal{Z}_h$. Then the normal form theorem applied to the reduced system for u_0 in the finite-dimensional subspace $\mathcal{E}_0$ shows that

$$u_0 = v_0 + \mathbf{\Phi}_\mu(v_0),$$

which leads to the normal form

$$\frac{dv_0}{dt} = \mathbf{L}_0 v_0 + \mathbf{N}_\mu(v_0) + \boldsymbol{\rho}(v_0, \mu). \tag{5.42}$$

Consequently, we can write

$$u = v_0 + \widetilde{\mathbf{\Psi}}(v_0, \mu), \tag{5.43}$$

with

$$\widetilde{\mathbf{\Psi}}(v_0, \mu) = \mathbf{\Phi}_\mu(v_0) + \mathbf{\Psi}(v_0 + \mathbf{\Phi}_\mu(v_0), \mu) \in \mathcal{Z}.$$

Notice that here $\widetilde{\mathbf{\Psi}}(v_0, \mu)$ belongs to the entire space $\mathcal{Z}$, and not to $\mathcal{Z}_h$ as $\mathbf{\Psi}(u_0, \mu)$. To obtain the normal form, we can now use the Ansatz (5.43), and proceed as for the algorithmic derivation in Section 5.2.3.

First, differentiating (5.43) with respect to t and replacing du/dt and dv_0/dt from (5.41) and (5.42), respectively, gives the identity

$$D_{v_0}\widetilde{\mathbf{\Psi}}(v_0, \mu)\mathbf{L}_0 v_0 - \mathbf{L}\widetilde{\mathbf{\Psi}}(v_0, \mu) + \mathbf{N}_\mu(v_0) = \mathbf{Q}(v_0, \mu), \tag{5.44}$$

where

$$\mathbf{Q}(v_0, \mu) = \mathbf{\Pi}_p \left(\mathbf{R}(v_0 + \widetilde{\mathbf{\Psi}}(v_0, \mu), \mu) - D_{v_0}\widetilde{\mathbf{\Psi}}(v_0, \mu)\mathbf{N}_\mu(v_0) \right).$$

Here $\mathbf{\Pi}_p$ represents the linear map that associates to a map of class $\mathcal{C}^p$ the polynomial of degree p in its Taylor expansion. Next, we set

$$\widetilde{\mathbf{\Psi}}(v_0, \mu) = \widetilde{\mathbf{\Psi}}_0(v_0, \mu) + \widetilde{\mathbf{\Psi}}_h(v_0, \mu),$$

where $\widetilde{\mathbf{\Psi}}_0 = \mathbf{P}_0\widetilde{\mathbf{\Psi}}$ and $\widetilde{\mathbf{\Psi}}_h = \mathbf{P}_h\widetilde{\mathbf{\Psi}}$ take values in $\mathcal{E}_0$ and $\mathcal{Z}_h$, respectively, according to the decomposition $\mathcal{Z} = \mathcal{E}_0 + \mathcal{Z}_h$. Projecting the identity (5.44) successively on $\mathcal{E}_0$ and $\mathcal{Z}_h$ with the projectors $\mathbf{P}_0$ and $\mathbf{P}_h$, respectively, gives the following system:

$$\mathcal{A}_{\mathbf{L}_0}\widetilde{\mathbf{\Psi}}_0(v_0, \mu) + \mathbf{N}_\mu(v_0) = \mathbf{Q}_0(v_0, \mu) \tag{5.45}$$

$$D_{v_0}\widetilde{\mathbf{\Psi}}_h(v_0, \mu)\mathbf{L}_0 v_0 - \mathbf{L}_h\widetilde{\mathbf{\Psi}}_h(v_0, \mu) = \mathbf{Q}_h(v_0, \mu), \tag{5.46}$$

where

$$\mathbf{Q}_0(v_0, \mu) = \mathbf{P}_0\mathbf{Q}(v_0, \mu), \quad \mathbf{Q}_h(v_0, \mu) = \mathbf{P}_h\mathbf{Q}.$$

We can solve both equations in this system using again the Taylor expansions of $\mathbf{R}$, $\widetilde{\mathbf{\Psi}}_0$, $\widetilde{\mathbf{\Psi}}_h$, and $\mathbf{N}_\mu$. Then equation (5.45) leads to an equation of the form (5.30), with $\mathbf{\Phi}_\mu(v)$ replaced by $\widetilde{\mathbf{\Psi}}_0(v_0, \mu)$ and can be solved as described in Section 5.2.3. Parallel to this, we have to solve the second equation, which determines $\widetilde{\mathbf{\Psi}}_h(v_0, \mu)$. This is also done with the help of the Taylor expansions, which lead at every order to an equation of the form

$$D_{v_0}\widetilde{\mathbf{\Psi}}_h(v_0)\mathbf{L}_0 v_0 - L_h\widetilde{\mathbf{\Psi}}_h(v_0) = \mathbf{Q}_h(v_0),$$

in which the right hand side is known. At this point we have to make sure that this equation has a solution $\widetilde{\mathbf{\Psi}}_h(v_0) \in \mathcal{Z}_h$. For this, notice that the equation above is obtained from the equation

$$\frac{d}{dt}\widetilde{\mathbf{\Psi}}_h(e^{\mathbf{L}_0 t}v_0) = \mathbf{L}_h\widetilde{\mathbf{\Psi}}_h(e^{\mathbf{L}_0 t}v_0) + \mathbf{Q}_h(e^{\mathbf{L}_0 t}v_0)$$

by taking $t = 0$. Here the map $t \mapsto \mathbf{Q}_h(e^{\mathbf{L}_0 t}v_0)$ belongs to $\mathcal{C}_\eta(\mathbb{R}, \mathcal{Y}_h)$ for any $\eta > 0$, so that thanks to Hypothesis (4.6) this equation possesses a unique solution $\mathbf{K}_h\mathbf{Q}_h(e^{\mathbf{L}_0 \cdot}v_0)$ (see [21] Appendix B2). Consequently, we may take

$$\widetilde{\boldsymbol{\Psi}}_h(v_0) = \left(\mathbf{K}_h\mathbf{Q}_h(e^{\mathbf{L}_0 \cdot}v_0)\right)(0),$$

which then shows that (5.46) can be solved at every order.

We show in the following sections, how to simultaneously compute the center manifold and the normal form for three different bifurcations in infinite-dimensional equations.

5.4.2 Example 1: Hopf Bifurcation

Consider an equation of the form (5.41), with a single parameter $\mu \in \mathbb{R}$, and satisfying the hypotheses in the center manifold Theorem 4.18. Further assume that the spectrum of the linear operator $\mathbf{L}$ contains precisely two purely imaginary eigenvalues $\pm i\omega$, which are simple.

Normal Form Under these assumptions, we have that $\sigma_0 = \{\pm i\omega\}$ and that the associated spectral subspace $\mathcal{E}_0$ is two-dimensional spanned by the eigenvectors ζ and $\overline{\zeta}$ associated with $i\omega$ and $-i\omega$, respectively. The center manifold Theorem 4.18, gives

$$u = u_0 + \boldsymbol{\Psi}(u_0, \mu), \quad u_0 \in \mathcal{E}_0, \quad \boldsymbol{\Psi}(u_0, \mu) \in \mathcal{Z}_h,$$

and applying the normal form Theorem 5.21 to the reduced system we find

$$u_0 = v_0 + \boldsymbol{\Phi}_\mu(v_0),$$

which gives the equality (5.43),

$$u \quad = \quad v_0 + \widetilde{\boldsymbol{\Psi}}(v_0, \mu), \quad v_0 \in \mathcal{E}_0, \quad \widetilde{\boldsymbol{\Psi}}(u_0, \mu) \in \mathcal{Z}.$$

For $v_0(t) \in \mathcal{E}_0$, it is convenient to write

$$v_0(t) \quad = \quad A(t)\zeta + \overline{A(t)\zeta}, \quad A(t) \in \mathbb{C},$$

and according to the Lemma 5.7 (see Remark 5.22(iv)), the polynomial $\mathbf{N}_\mu(A, \overline{A})$ in the normal form is of the form

$$\mathbf{N}_\mu(A, \overline{A}) = (AQ(|A|^2, \mu), \overline{AQ}(|A|^2, \mu)),$$

with Q a complex-valued polynomial in its first argument satisfying $Q(0, 0) = 0$.

Computation of the Normal Form Our purpose it to show how to compute the two leading order coefficients in the expression of $\mathbf{N}_\mu$, i.e., the coefficients a and b in the expression

$$Q(|A|^2, \mu) = a\mu + b|A|^2 + O((|\mu| + |A|^2)^2).$$

For this calculation we proceed as indicated in Section 5.4.

We start from the identity (5.44) in which we replace the Taylor expansions of $\mathbf{R}$ and $\widetilde{\mathbf{\Psi}}$. With the notations from Section 5.2.3, we set

$$\widetilde{\mathbf{\Psi}}_{ql}(v_0^{(q)}, \mu^{(l)}) = \mu^l \sum_{q_1+q_2=q} A^{q_1}\overline{A}^{q_2}\mathbf{\Psi}_{q_1q_2l}, \quad \mathbf{\Psi}_{q_1q_2l} \in \mathcal{Z}.$$

By identifying in (5.44) the terms of order $O(\mu)$, $O(A^2)$, and $O(A\overline{A})$, we obtain

$$\begin{aligned} -\mathbf{L}\mathbf{\Psi}_{001} &= \mathbf{R}_{01}, \\ (2i\omega - \mathbf{L})\mathbf{\Psi}_{200} &= \mathbf{R}_{20}(\zeta, \zeta), \\ -\mathbf{L}\mathbf{\Psi}_{110} &= 2\mathbf{R}_{20}(\zeta, \overline{\zeta}). \end{aligned}$$

Here the operators $\mathbf{L}$ and $(2i\omega - \mathbf{L})$ on the left hand sides are invertible, so that $\mathbf{\Psi}_{001}$, $\mathbf{\Psi}_{200}$, and $\mathbf{\Psi}_{110}$ are uniquely determined from these equalities. Next, we identify the terms of order $O(\mu A)$ and $O(A^2\overline{A})$ and find

$$\begin{aligned} (i\omega - \mathbf{L})\mathbf{\Psi}_{101} &= -a\zeta + \mathbf{R}_{11}(\zeta) + 2\mathbf{R}_{20}(\zeta, \mathbf{\Psi}_{001}), \\ (i\omega - \mathbf{L})\mathbf{\Psi}_{210} &= -b\zeta + 2\mathbf{R}_{20}(\zeta, \mathbf{\Psi}_{110}) + 2\mathbf{R}_{20}(\overline{\zeta}, \mathbf{\Psi}_{200}) + 3\mathbf{R}_{30}(\zeta, \zeta, \overline{\zeta}). \end{aligned}$$

Since $i\omega$ is a simple isolated eigenvalue of $\mathbf{L}$, the range of $(i\omega - \mathbf{L})$ is of codimension 1, so that we can solve these equations and determine $\mathbf{\Psi}_{101}$ and $\mathbf{\Psi}_{200}$, provided the right hand sides satisfy one solvability condition. It is this solvability condition which allows us to compute the coefficients a and b, just in the finite-dimensional case. In the case where $\mathbf{L}$ has an adjoint $\mathbf{L}^*$ acting in the dual space $\mathcal{X}^*$, the solvability condition is that the right hand sides be orthogonal to the kernel of the adjoint $(-i\omega - \mathbf{L}^*)$ of $(i\omega - \mathbf{L})$. The kernel of $(-i\omega - \mathbf{L}^*)$ is one-dimensional, just as the kernel of $(i\omega - \mathbf{L})$, spanned by $\zeta^* \in \mathcal{X}^*$ that we choose such that $\langle \zeta, \zeta^* \rangle = 1$. Here $\langle \cdot, \cdot \rangle$ denotes the duality product between $\mathcal{X}$ and $\mathcal{X}^*$, where it is semilinear with respect to the second argument. Then in this situation we find

$$\begin{aligned} a &= \langle \mathbf{R}_{11}(\zeta) + 2\mathbf{R}_{20}(\zeta, \mathbf{\Psi}_{001}), \zeta^* \rangle, \\ b &= \langle 2\mathbf{R}_{20}(\zeta, \mathbf{\Psi}_{110}) + 2\mathbf{R}_{20}(\overline{\zeta}, \mathbf{\Psi}_{200}) + 3\mathbf{R}_{30}(\zeta, \zeta, \overline{\zeta}), \zeta^* \rangle. \end{aligned}$$

Notice that here it is not necessary to further solve the equations and compute $\mathbf{\Psi}_{101}$ and $\mathbf{\Psi}_{210}$.

Now, if the adjoint $\mathbf{L}^*$ does not exist, we still have a Fredholm alternative for the equations above. Indeed, both equations are of the form

$$(i\omega - \mathbf{L})\mathbf{\Psi} = \mathbf{R}, \tag{5.47}$$

with $\mathbf{R} \in \mathcal{X}$. Projecting with $\mathbf{P}_0$ and $\mathbf{P}_h$ on the subspaces $\mathcal{E}_0$ and $\mathcal{X}_h$, respectively, we obtain

$$\begin{aligned} (i\omega - \mathbf{L}_0)\mathbf{P}_0\mathbf{\Psi} &= \mathbf{P}_0\mathbf{R}, \\ (i\omega - \mathbf{L}_h)\mathbf{P}_h\mathbf{\Psi} &= \mathbf{P}_h\mathbf{R}. \end{aligned}$$

The operator on the left hand side of the second equation is invertible, since the spectrum of $\mathbf{L}_h$ is $\sigma_- \cup \sigma_+$, which is bounded away from the imaginary axis (see Hypothesis 4.4). Then the second equation has a unique solution

$$\mathbf{P}_h\mathbf{\Psi} = (i\omega - \mathbf{L}_h)^{-1}\mathbf{P}_h\mathbf{R}, \quad (i\omega - \mathbf{L}_h)^{-1} : \mathcal{X}_h \to \mathcal{Z}_h.$$

The first equation is two-dimensional, so that there is a solution $\mathbf{\Psi}_0$, provided the following solvability condition holds

$$\langle \mathbf{R}_0, \zeta_0^* \rangle = 0,$$

where $\zeta_0^* \in \mathcal{E}_0$ is the eigenvector in the kernel of the adjoint $(-i\omega - \mathbf{L}_0^*)$ in $\mathcal{E}_0$ chosen such that $\langle \zeta, \zeta_0^* \rangle = 1$. We rewrite this solvability condition as

$$\langle \mathbf{R}_0, \zeta_0^* \rangle = \langle \mathbf{P}_0 \mathbf{R}, \zeta_0^* \rangle = \langle \mathbf{R}, \mathbf{P}_0^* \zeta_0^* \rangle = 0, \tag{5.48}$$

in which $\mathbf{P}_0^*$ is the adjoint of the projector $\mathbf{P}_0$, and the last bracket represents the duality product between $\mathcal{X}$ and $\mathcal{X}^*$. Upon setting

$$\zeta^* = \mathbf{P}_0^* \zeta_0^* \in \mathcal{X}^*,$$

the solvability condition becomes $\langle \mathbf{R}, \zeta^* \rangle = 0$, which then leads to formulas for the coefficients a and b as above.

We point out that the range of $i\omega - \mathbf{L}$ is orthogonal to the vector ζ^* constructed above, with respect to the duality product between $\mathcal{X}$ and $\mathcal{X}^*$, and actually, its range is precisely the space orthogonal to ζ^*. Indeed, since $i\omega$ is an isolated simple eigenvalue of $\mathbf{L}$, the operator is Fredholm with index zero, so its range is closed and has a codimension equal to the dimension of the kernel, which is 1.

Reduced Dynamics The dynamics of the reduced equation, which is two-dimensional, is as described in Theorem 3.6, so that we are here in the presence of a Hopf bifurcation. We then have a branch of equilibria for small μ and a family of periodic solutions of size $O(|\mu|^{1/2})$, which bifurcate at $\mu = 0$ for μ such that $a_r b_r \mu < 0$. Here a_r and b_r denote the real parts of a and b, respectively.

We point out that in this situation, the stability of both equilibria and periodic solutions is the same in the reduced system and in the full equation. Indeed, for all these solutions, one has a strong stable manifold of codimension 2 corresponding to perturbations of the stable spectrum σ_- of $\mathbf{L}$, and the remaining dynamics are found on the center manifold. For example, assume that $a_r > 0$. Then the family of equilibria is stable for $\mu < 0$ and loses its stability when μ crosses 0 (see Theorem 3.6). In the supercritical case, when $b_r < 0$, we have an *attracting periodic solution on the center manifold for* $\mu > 0$, for which we can compute the Floquet exponents. The most unstable exponents correspond to the flow on the center manifold, which give here 0, due to the invariance under translations in time t of (5.41), and a real negative exponent, close to 0. The other exponents correspond to perturbations of the stable eigenvalues in σ_- of $\mathbf{L}$, and give a strong stable manifold of codimension 2, transverse to the weakly stable mode obtained from the dynamics on the center manifold. It results that in the supercritical case the bifurcating periodic solution is also stable in $\mathcal{Y}$. In the subcritical case, when $b_r > 0$, the periodic solution occurs for $\mu < 0$ and is unstable, since it is already unstable on the center manifold.

5.4.3 Example 2: Elliptic PDE in a strip

We come back to the example treated at subsection 4.4.3 in which we take

$$g(u_1, u_2, u_3) = -\alpha u_1^2 - \beta u_1 u_3 - \gamma u_2^2 - \delta u_3^2.$$

We saw in Section 4.4.3 that the problem can be written in the form (4.10), to which we can apply Theorems 4.18 and 4.31. The resulting two-dimensional reduced system (4.49) is reversible and satisfies the hypotheses in Theorems 5.21 and 5.27, so that we can choose coordinates (A, B) in $\mathbb{R}^2$ such that the center manifold reads

$$u = A\zeta_0 + B\zeta_1 + \widetilde{\mathbf{\Psi}}(A, B, \mu)$$

and the reduced system is in normal form

$$\begin{aligned} \frac{dA}{dx} &= B \\ \frac{dB}{dx} &= Q(A, \mu) + \rho(A, B^2, \mu), \end{aligned} \tag{5.49}$$

with $\rho(A, B^2, \mu) = O((|A| + |B|)^p)$. Notice that here 0 is a solution of (4.10) for all μ, so that $Q(0, \mu) = 0$, which then leads to the expansion

$$Q(A, \mu) = c\mu A + bA^2 + O(|\mu|^2|A| + |A|^3).$$

Furthermore the coefficients in the expansion of $\widetilde{\mathbf{\Psi}}$ now satisfy

$$\begin{aligned} \widetilde{\mathbf{\Psi}}(A, B, \mu) &= \sum_{1\leq r+s+q\leq p} \mathbf{\Psi}_{rsq} A^r B^s \mu^q + o((|A| + |B| + |\mu|)^p), \\ & \mathbf{\Psi}_{100} = \mathbf{\Psi}_{010} = 0. \end{aligned} \tag{5.50}$$

hence

$$\mathbf{\Psi}_{00q} = 0, \quad \mathbf{S}\mathbf{\Psi}_{rsq} = (-1)^s \mathbf{\Psi}_{rsq}.$$

In order to compute the coefficients c and b we consider the identity:

$$(\partial_A \widetilde{\mathbf{\Psi}})B + (\zeta_1 + \partial_B \widetilde{\mathbf{\Psi}})Q(A, \mu) = \mathbf{L}\widetilde{\mathbf{\Psi}} + \mathbf{R}(A\zeta_0 + B\zeta_1 + \widetilde{\mathbf{\Psi}}, \mu), \tag{5.51}$$

and identify powers of (A, B, μ) in (5.51). At orders μA and A^2 we find

$$c\zeta_1 = \mathbf{L}\mathbf{\Psi}_{101} + \mathbf{R}_{1,1}\zeta_0 \tag{5.52}$$

$$b\zeta_1 = \mathbf{L}\mathbf{\Psi}_{200} + \mathbf{R}_{2,0}(\zeta_0, \zeta_0), \tag{5.53}$$

so that c and b can be found from the compatibility conditions for these equations:

$$\begin{aligned} c &= \langle \mathbf{R}_{1,1}\zeta_0, \zeta_1^* \rangle \\ b &= \langle \mathbf{R}_{2,0}(\zeta_0, \zeta_0), \zeta_1^* \rangle. \end{aligned}$$

In this case the operator $\mathbf{L}$ has a well-defined adjoint $\mathbf{L}^*$ so that ζ_1^* is the vector in the kernel of $\mathbf{L}^*$ satisfying $\langle \zeta_1, \zeta_1^* \rangle = 1$. Using the explicit formulas of the different terms in these equalities, a direct calculation gives

$$c = -1, \quad b = \frac{2}{3\pi}(4\alpha + 2\delta).$$

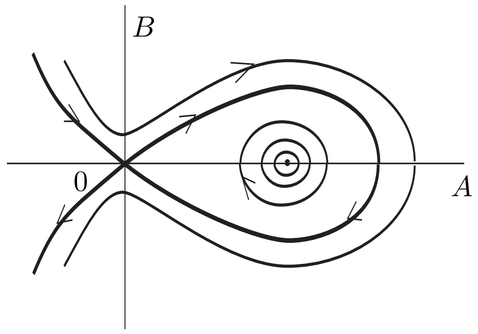


Figure 5.2: Dynamics in the case $b < 0, \mu < 0$.

Assuming that $b \neq 0$ and $\mu \neq 0$, the phase portrait of this system is indicated on Figure 5.2 (case $\mu < 0, b < 0$). Other cases are similar and easy to derive. In particular, there is a homoclinic orbit to the saddle equilibrium, and the center equilibrium is surrounded by a family of periodic orbits. In the full elliptic PDE, the homoclinic orbit corresponds to an asymptotically homogeneous solution $v(x, y)$, which tends as $x \to \pm\infty$ towards the same x-independent solution $v_*(y)$. The periodic orbits correspond to a family of solutions of the PDE, which are periodic in x and which tend to the asymptotically homogeneous solution, as their period tends to infinity.

Remark 5.29. *(i) Alternatively, for the computation of the coefficient c we can use the result in the Exercise 4.20. This implies that the two eigenvalues of the linearization of the normal form (5.49) at 0 for small μ are precisely the two eigenvalues of the linearization $\mathbf{L} + D\mathbf{R}(0, \mu)$ that vanish at $\mu = 0$. According to the formula (4.48), these two eigenvalues are $\pm\sqrt{-\mu}$, which then gives $c = -1$.*

(ii) The equality (5.53) leads to the second order differential equation

$$u_{200}'' + u_{200} + b \sin y = \frac{\alpha + \delta}{2} - \frac{\alpha - \delta}{2} \cos(2y) + \frac{\beta}{2} \sin(2y),$$

in which u_{200} represents the first component of the vector $\mathbf{\Psi}_{200}$ and satisfies $u_{200}|_{y=0} = u_{200}|_{y=\pi} = 0$. Taking the scalar product of this equality with $\sin y$ and integrating by parts gives the formula for b above.

5.4.4 Example 3: Hopf Bifurcations with Symmetries

We discuss in this section two examples of Hopf bifurcations, with symmetries $SO(2)$ and $O(2)$. While in the first case the symmetry implies that the reduced system is always in normal form, in the second case we apply the result in Theorem 5.25 to determine the normal form of the reduced system.

Hopf Bifurcation with $SO(2)$ Symmetry

Consider the situation in Section 5.4.2 of an equation of the form (5.41), with a single parameter $\mu \in \mathbb{R}$, satisfying the hypotheses in center manifold Theorem 4.18 , and such that the spectrum of the linear operator $\mathbf{L}$ contains precisely two purely imaginary eigenvalues $\pm i\omega$, which are simple. We now further assume that there is a one-parameter continuous family of linear maps $\mathbf{R}_\varphi \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Z})$ for $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, with the following properties:

(i) $\mathbf{R}_\varphi \circ \mathbf{R}_\psi = \mathbf{R}_{\varphi+\psi}$ for all $\varphi,\ \psi \in \mathbb{R}/2\pi\mathbb{Z}$;

(ii) $\mathbf{R}_0 = \mathbb{I}$;

(iii) $\mathbf{R}_\varphi \mathbf{L} = \mathbf{L}\mathbf{R}_\varphi$ and $\mathbf{R}(\mathbf{R}_\varphi u, \mu) = \mathbf{R}_\varphi \mathbf{R}(u, \mu)$ for all $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, $u \in \mathcal{Z}$, and $\mu \in \mathbb{R}$.

In particular, the group $\{\mathbf{R}_\varphi; \varphi \in \mathbb{R}/2\pi\mathbb{Z}\}$ is a representation of an $SO(2)$ symmetry in $\mathcal{X}$ and $\mathcal{Z}$. As in the two-dimensional case discussed in Section 3.1, these properties allow us to simplify the analysis of the reduced equation, and induce some symmetry properties for the bifurcating periodic solutions.

Reduced System Consider the eigenvector ζ associated to the simple eigenvalue $i\omega$ of $\mathbf{L}$. Then, by arguing as in Section 3.1 from the fact that $\mathbf{R}_\varphi$ commutes with $\mathbf{L}$, we find that

$$\mathbf{R}_\varphi \zeta = e^{im\varphi}\zeta,$$

for some $m \in \mathbb{Z}$. In the case $m = 0$, which means that the action of all $\mathbf{R}_\varphi$ on the subspace $\mathcal{E}_0$ is trivial, the results in Section 5.4.2 hold with the additional property that the periodic solution is pointwise invariant under the "rotations" $\mathbf{R}_\varphi$.

Assume that $m \neq 0$. Then we choose a norm on $\mathcal{E}_0$ such that $\mathbf{R}_\varphi$ is an isometry, and applying the result in Theorem 4.29, we find that the reduction function $\mathbf{\Psi}$ satisfies

$$\mathbf{R}_\varphi \mathbf{\Psi}(u_0, \mu) = \mathbf{\Psi}(\mathbf{R}_\varphi u_0, \mu) \text{ for all } u_0 \in \mathcal{E}_0,\ \mu \in \mathbb{R}.$$

We set again

$$u_0(t) = A(t)\zeta + \overline{A(t)\zeta}$$

for $u_0(t) \in \mathcal{E}_0$, with A a complex-valued function, and then the reduced system is

$$\frac{dA}{dt} = i\omega A + f(A, \overline{A}, \mu),$$

together with the complex conjugated equation. In addition, the vector field commutes with $\mathbf{R}_\varphi|_{\mathcal{E}_0}$, which together with the equality $\mathbf{R}_\varphi \zeta = e^{im\varphi}\zeta$ implies that

$$f(e^{im\varphi}A, e^{-im\varphi}\overline{A}, \mu) = e^{im\varphi} f(A, \overline{A}, \mu).$$

According to Lemma 3.4, we then have that

$$f(A, \overline{A}, \mu) = Ag(|A|^2, \mu),$$

with g of class C^{k-1}, so that in this case the reduced equation in already in normal form.

Reduced Dynamics This situation was discussed in Section 3.1. First, $A = 0$ is always an equilibrium of the reduced system, which gives the equilibria $u = \Psi(0, \mu)$. These equilibria are invariant under the action of $\mathbf{R}_\varphi$. Next, according to the results in Corollary 3.12, the reduced equation possesses a family of periodic solutions

$$A(t, \mu) = r(\mu)e^{i\omega_*(\mu)t}, \quad r(\mu) = O(|\mu|^{1/2}),$$

which are rotating waves, with

$$A(t, \mu) = \mathbf{R}_{\frac{\omega_*(\mu)t}{m}} A(0, \mu)$$

satisfying

$$\mathbf{R}_\varphi A(t, \mu)\zeta = A(t + \frac{m\varphi}{\omega_*}, \mu)\zeta.$$

Using the fact that $\mathbf{R}_\varphi$ commutes with the reduction function $\mathbf{\Psi}$, we find that the corresponding solutions $u(\cdot, \mu)$ of the full equation satisfy

$$\begin{aligned} \mathbf{R}_\varphi u(t, \mu) &= \mathbf{R}_\varphi(u_0(t, \mu) + \mathbf{\Psi}(u_0(t, \mu), \mu)) \\ &= u_0(t + \frac{m\varphi}{\omega_*}, \mu) + \mathbf{\Psi}(u_0(t + \frac{m\varphi}{\omega_*}, \mu), \mu) = u(t + \frac{m\varphi}{\omega_*}, \mu). \end{aligned}$$

By arguing as for (3.38), this implies that $u(\cdot, \mu)$ is also *a rotating wave*, i.e.,

$$u(t, \mu) = \mathbf{R}_{\frac{\omega_*(\mu)t}{m}} u(0, \mu). \tag{5.54}$$

Hopf Bifurcation with $O(2)$ Symmetry

In the same setting as above, we now further assume that there exists a symmetry $\mathbf{S}$, with $\mathbf{S}^2 = \mathbb{I}$, such that the vector field is equivariant under the action of $\mathbf{S}$,

$$\mathbf{SL} = \mathbf{LS}, \quad \mathbf{R}(\mathbf{S}u, \mu) = \mathbf{SR}(u, \mu) \text{ for all } \mu \in \mathbb{R} \tag{5.55}$$

and that

$$\mathbf{R}_\varphi \mathbf{S} \;=\; \mathbf{SR}_{-\varphi} \text{ for all } \varphi \in \mathbb{R}/2\pi\mathbb{Z}. \tag{5.56}$$

Then the group $\{\mathbf{R}_\varphi, \mathbf{S}; \varphi \in \mathbb{R}/2\pi\mathbb{Z}\}$ is a representation of an $O(2)$ symmetry in $\mathcal{X}$ and $\mathcal{Z}$. We already met this type of symmetry in Section 3.4 and in the example in Section 4.4.2.

A key property here is that generically the eigenvalues of the linear operator $\mathbf{L}$ are at least geometrically double. Indeed, by arguing as in Section 3.4, one concludes that any eigenvalue λ of $\mathbf{L}$ that has an eigenvector ζ which is not invariant under the action of $\mathbf{R}_\varphi$ (i.e., $\mathbf{R}_\varphi\zeta \neq \zeta$ for some $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$) is at least geometrically double. We shall therefore assume in this example that $\sigma_0 = \{\pm i\omega\}$, where $\pm i\omega$ are algebraically and geometrically double eigenvalues, with associated eigenvectors that are not invariant under the action of $\mathbf{R}_\varphi$. Then the restriction of the action of $\mathbf{R}_\varphi$ to the eigenspaces associated with the eigenvalues $\pm i\omega$ is not trivial, and the result in (3.39) shows that we can choose the eigenvectors $\{\zeta_0, \zeta_1\}$ associated with $i\omega$ such that

$$\mathbf{R}_\varphi\zeta_0 = e^{im\varphi}\zeta_0, \quad \mathbf{R}_\varphi\zeta_1 = e^{-im\varphi}\zeta_1, \quad \mathbf{S}\zeta_0 = \zeta_1, \quad \mathbf{S}\zeta_1 = \zeta_0. \tag{5.57}$$

Clearly, $\{\overline{\zeta}_0, \overline{\zeta}_1\}$ are the eigenvectors associated with $-i\omega$.

Normal Form We can now choose a norm on $\mathcal{E}_0$ such that $\mathbf{R}_\varphi$ and $\mathbf{S}$ are isometries, and applying the result in Theorem 4.29, we find that the reduction function $\mathbf{\Psi}$ satisfies

$$\mathbf{\Psi}(\mathbf{R}_\varphi u_0,\mu) = \mathbf{R}_\varphi \mathbf{\Psi}(u_0,\mu), \quad \mathbf{\Psi}(\mathbf{S}u_0,\mu) = \mathbf{S}\mathbf{\Psi}(u_0,\mu) \text{ for all } u_0 \in \mathcal{E}_0,\ \mu \in \mathbb{R}.$$

Further applying Theorems 5.21 and 5.25 to the reduced equation, we write

$$u = v_0 + \widetilde{\mathbf{\Psi}}(v_0,\mu), \quad v_0 \in \mathcal{E}_0, \quad \widetilde{\mathbf{\Psi}}(v_0,\mu) \in \mathcal{Z},$$

and set

$$v_0(t) = A(t)\zeta_0 + B(t)\zeta_1 + \overline{A(t)\zeta_0} + \overline{B(t)\zeta_1}.$$

Here A and B are complex-valued functions, and $\widetilde{\mathbf{\Psi}}(\cdot,\mu)$ commutes with $\mathbf{R}_\varphi$ and $\mathbf{S}$.

The polynomial $\mathbf{N}_\mu$ in the resulting normal form satisfies the characterization (5.28) and also commutes with $\mathbf{R}_\varphi$ and $\mathbf{S}$. We write

$$\mathbf{N}_\mu = (\Phi_0, \Phi_1, \overline{\Phi}_0, \overline{\Phi}_1)$$

where Φ_j, $j = 0,1$, are polynomials of $(A, B, \overline{A}, \overline{B})$ with coefficients depending upon μ. Using successively the characterization (5.28) and the fact that $\mathbf{N}_\mu$ commutes with $\mathbf{R}_\varphi$ and $\mathbf{S}$, we find that

$$\begin{aligned}
\Phi_0(e^{-i\omega t}A, e^{-i\omega t}B, e^{i\omega t}\overline{A}, e^{i\omega t}\overline{B}) &= e^{-i\omega t}\Phi_0(A,B,\overline{A},\overline{B}),\\
\Phi_1(e^{-i\omega t}A, e^{-i\omega t}B, e^{i\omega t}\overline{A}, e^{i\omega t}\overline{B}) &= e^{-i\omega t}\Phi_1(A,B,\overline{A},\overline{B}),\\
\Phi_0(e^{im\varphi}A, e^{-im\varphi}B, e^{-im\varphi}\overline{A}, e^{im\varphi}\overline{B}) &= e^{im\varphi}\Phi_0(A,B,\overline{A},\overline{B}),\\
\Phi_1(e^{im\varphi}A, e^{-im\varphi}B, e^{-im\varphi}\overline{A}, e^{im\varphi}\overline{B}) &= e^{-im\varphi}\Phi_1(A,B,\overline{A},\overline{B}),\\
\Phi_0(B, A, \overline{B}, \overline{A}) &= \Phi_1(A,B,\overline{A},\overline{B})
\end{aligned} \tag{5.58}$$

for all $t \in \mathbb{R}$ and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$.

To exploit these identities we proceed as follows. The first and third identities lead to

$$\Phi_0(e^{i(m\varphi-\omega t)}A, e^{-i(m\varphi+\omega t)}B, e^{i(\omega t-m\varphi)}\overline{A}, e^{i(m\varphi+\omega t)}\overline{B}) = e^{i(m\varphi-\omega t)}\Phi_0(A,B,\overline{A},\overline{B})$$

for any $t \in \mathbb{R}$ and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$. We choose (t,φ) such that

$$m\varphi - \omega t = -\arg A, \quad m\varphi + \omega t = \arg B,$$

which implies that

$$\Phi_0(A,B,\overline{A},\overline{B}) = e^{i\arg A}\Phi_0(|A|,|B|,|A|,|B|).$$

Then we choose (t,φ) such that

$$m\varphi - \omega t = \pi, \quad m\varphi + \omega t = 0,$$

which gives

$$\Phi_0(-A, B, -\overline{A}, \overline{B}) = -\Phi_0(A,B,\overline{A},\overline{B}),$$

and finally we choose (t, φ) such that

$$m\varphi - \omega t = 0, \quad m\varphi + \omega t = \pi,$$

which shows that

$$\Phi_0(A, -B, \overline{A}, -\overline{B}) = \Phi_0(A, B, \overline{A}, \overline{B}).$$

Since Φ_0 is a polynomial, it follows now that there is a polynomial P_0 such that

$$\Phi_0(A, B, \overline{A}, \overline{B}) = AP_0(|A|^2, |B|^2),$$

and similarly we obtain that there is a polynomial P_1 such that

$$\Phi_1(A, B, \overline{A}, \overline{B}) = BP_1(|A|^2, |B|^2).$$

In addition, from the last identity in (5.58) we conclude that

$$P_1(|A|^2, |B|^2) = P_0(|B|^2, |A|^2).$$

Summarizing, we have the normal form

$$\begin{aligned} \frac{dA}{dt} &= i\omega A + AP(|A|^2, |B|^2, \mu) + \rho(A, B, \overline{A}, \overline{B}, \mu) \\ \frac{dB}{dt} &= i\omega B + BP(|B|^2, |A|^2, \mu) + \rho(B, A, \overline{B}, \overline{A}, \mu), \end{aligned} \tag{5.59}$$

in which P is a polynomial of degree p in its first two arguments with coefficients depending upon μ, as given in Theorem 5.21, and $\rho(A, B, \overline{A}, \overline{B}, \mu) = O((|A| + |B|)^{2p+3})$. Furthermore, notice here the particular form of the remainder ρ, which is due to the fact that the vector field in this system commutes with $\mathbf{S}$, whereas from the fact that the vector field commutes with $\mathbf{R}_\varphi$ we have in addition that

$$\rho(e^{im\varphi}A, e^{im\varphi}B, e^{-im\varphi}\overline{A}, e^{-im\varphi}\overline{B}, \mu) \quad = \quad e^{im\varphi}\rho(A, B, \overline{A}, \overline{B}, \mu).$$

Exercise 5.30 (Computation of the normal form)**.** *Consider the normal form truncated at order 3,*

$$\begin{aligned} \frac{dA}{dt} &= i\omega A + A(a\mu + b|A|^2 + c|B|^2), \\ \frac{dB}{dt} &= i\omega B + B(a\mu + b|B|^2 + c|A|^2), \end{aligned}$$

with complex coefficients a, b, and c, and the Taylor expansion of $\widetilde{\mathbf{\Psi}}$*,*

$$\widetilde{\mathbf{\Psi}}(A, B, \overline{A}, \overline{B}, \mu) \quad = \sum_{p+q+r+s+l \geq 1} \mathbf{\Psi}_{pqrsl} A^p \overline{A}^q B^r \overline{B}^s \mu^l,$$

in which $\mathbf{\Psi}_{10000} = \mathbf{\Psi}_{01000} = \mathbf{\Psi}_{00100} = \mathbf{\Psi}_{00010} = 0$*. Show that*

$$\begin{aligned} &\mathbf{\Psi}_{00001} = -\mathbf{L}^{-1}\mathbf{R}_{01}, \quad \mathbf{\Psi}_{20000} = (2i\omega - \mathbf{L})^{-1}\mathbf{R}_{20}(\zeta_0, \zeta_0), \\ &\mathbf{\Psi}_{11000} = -2\mathbf{L}^{-1}\mathbf{R}_{20}(\zeta_0, \overline{\zeta}_0), \quad \mathbf{\Psi}_{00110} = \mathbf{S}\mathbf{\Psi}_{11000}, \\ &\mathbf{\Psi}_{10100} = 2(2i\omega - \mathbf{L})^{-1}\mathbf{R}_{20}(\zeta_0, \zeta_1), \quad \mathbf{\Psi}_{10010} = -2\mathbf{L}^{-1}\mathbf{R}_{20}(\zeta_0, \overline{\zeta}_1), \end{aligned}$$

and that the coefficients a, b, c are given by

$$\begin{aligned} a &= \langle \mathbf{R}_{11}(\zeta_0) + 2\mathbf{R}_{20}(\zeta_0, \mathbf{\Psi}_{00001}), \zeta_0^* \rangle, \\ b &= \langle 2\mathbf{R}_{20}(\zeta_0, \mathbf{\Psi}_{11000}) + 2\mathbf{R}_{20}(\overline{\zeta}_0, \mathbf{\Psi}_{20000}) + 3\mathbf{R}_{30}(\zeta_0, \zeta_0, \overline{\zeta}_0), \zeta_0^* \rangle, \\ c &= \langle 2\mathbf{R}_{20}(\zeta_0, \mathbf{\Psi}_{00110}) + 2\mathbf{R}_{20}(\zeta_1, \mathbf{\Psi}_{10010}) + 2\mathbf{R}_{20}(\overline{\zeta}_1, \mathbf{\Psi}_{10100}) + 6\mathbf{R}_{30}(\zeta_0, \zeta_1, \overline{\zeta}_1), \zeta_0^* \rangle, \end{aligned}$$

where $\zeta_0^* \in \mathcal{X}^*$ is constructed as ζ^* in Section 5.4.2.

Reduced Dynamics The study of the dynamics of the system (5.59) strongly relies upon the study of the normal form truncated at order 3. In polar coordinates

$$A = r_0 e^{i\theta_0}, \quad B = r_1 e^{i\theta_1},$$

the truncated normal form becomes

$$\begin{aligned} \frac{dr_0}{dt} &= r_0(a_r\mu + b_r r_0^2 + c_r r_1^2), \\ \frac{dr_1}{dt} &= r_1(a_r\mu + b_r r_1^2 + c_r r_0^2), \\ \frac{d\theta_0}{dt} &= \omega + a_i\mu + b_i r_0^2 + c_i r_1^2, \\ \frac{d\theta_1}{dt} &= \omega + a_i\mu + b_i r_1^2 + c_i r_0^2, \end{aligned} \tag{5.60}$$

where the subscripts r and i indicate the real and the imaginary parts, respectively, of a complex number. Here the two first equations for (r_0, r_1) decouple from the last two equations for the phases (θ_0, θ_1), so that we can solve them separately.

The dynamics of these two equations are rather simple and are summarized in the case $a_r\mu > 0$ in Figure 5.3. (Similar phase portraits can be found in the other cases.) In

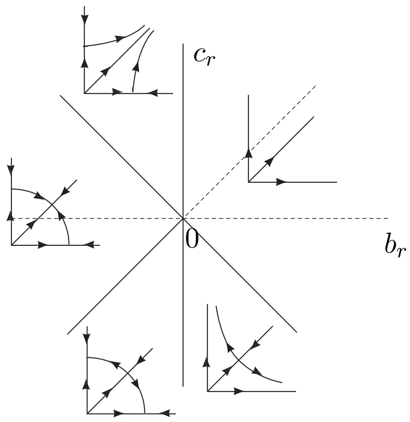


Figure 5.3: Phase portraits in the (r_0, r_1)-plane of the equations for (r_0, r_1), depending upon (b_r, c_r) in the case $a_r\mu > 0$.

particular, for $b_r < 0$ in this case, one finds two pair of equilibria $(\pm r_*(\mu), 0)$ and $(0, \pm r_*(\mu))$

on the r_0- and r_1-axis, respectively. These equilibria correspond to *rotating waves*, just as for the Hopf bifurcation in the presence of $SO(2)$ symmetry. Here, the symmetry $\mathbf{S}$ exchanges the two axes, so that it exchanges the rotating waves corresponding to $r_0 = 0$ into the rotating waves corresponding to $r_1 = 0$. Their stability is indicated in Figure 5.3, and we refer for instance to [26] for a proof of the persistence of these rotating waves for the full system (5.59). Next, for $b_r + c_r < 0$ in this case, there is another pair of equilibria with $r_0 = r_1$, which correspond to *standing waves*, another class of bifurcating periodic solutions (e.g., see [26] for a proof of the persistence of these solutions for (5.59)). These correspond to a torus of solutions of the normal form

$$\begin{aligned} v_0(t,\mu,\delta_0,\delta_1) &= r_0(\mu)\left(e^{i(\omega_*(\mu)t+\delta_0)}\zeta_0 + e^{i(\omega_*(\mu)t+\delta_1)}\zeta_1\right) \\ &+ r_0(\mu)\left(e^{-i(\omega_*(\mu)t+\delta_0)}\overline{\zeta}_0 + e^{-i(\omega_*(\mu)t+\delta_1)}\overline{\zeta}_1\right) \end{aligned}$$

for any $(\delta_0, \delta_1) \in \mathbb{R}^2$, which induces a torus of solutions $u(t,\mu,\delta_0,\delta_1)$ in $\mathcal{Y}$ of the full system (5.41). Notice that these standing waves possess the following symmetry properties:

$$\begin{aligned} &\mathbf{R}_{\frac{\delta_1-\delta_0}{m}}\mathbf{S}u(t,\mu,\delta_0,\delta_1) = u(t,\mu,\delta_0,\delta_1), \quad \mathbf{R}_{\frac{2\pi}{m}}u(t,\mu,\delta_0,\delta_1) = u(t,\mu,\delta_0,\delta_1), \\ &\mathbf{R}_{\frac{\pi}{m}}u(t,\mu,\delta_0,\delta_1) = u(t+\frac{\pi}{\omega_*(\mu)},\mu,\delta_0,\delta_1), \quad \mathbf{S}u(t,\mu,\delta_0,\delta_0) = u(t,\mu,\delta_0,\delta_0). \end{aligned}$$

Exercise 5.31. *Consider a system of the form (5.41) with $\mu \in \mathbb{R}^2$ satisfying the hypotheses of Theorem 4.18. Further assume that*

(i) the linear operator $\mathbf{L}$ has precisely three eigenvalues on the imaginary axis, $\sigma_0 = \{\pm i\omega, 0\}$, which are all simple;

(ii) $\mathbf{L}$ and $\mathbf{R}(\cdot,\mu)$ commute with a symmetry $\mathbf{S}$, with $\mathbf{S}^2 = \mathbb{I}$;

(iii) the eigenvector ζ associated with the eigenvalue 0 is antisymmetric, $\mathbf{S}\zeta = -\zeta$.

Using the result in Lemma 5.12, derive the normal form for the three-dimensional reduced system, and give formulas for the coefficients of the linear and cubic terms. (The study of the dynamics of the reduced vector field in this situation can be found in [48].)

5.4.5 Example 4: Takens–Bogdanov Bifurcation

Consider now an equation of the form (5.41), with a parameter $\mu \in \mathbb{R}^m$, and satisfying the hypotheses in the center manifold Theorem 4.18. Further assume that 0 is the only eigenvalue of $\mathbf{L}$ on the imaginary axis and that this eigenvalue is geometrically simple and algebraically double.

Normal Form With these assumptions we have $\sigma_0 = \{0\}$, and the associated spectral subspace $\mathcal{E}_0$ is two-dimensional. We choose a basis $\{\zeta_0, \zeta_1\}$ in $\mathcal{E}_0$ such that

$$\mathbf{L}\zeta_0 = 0, \quad \mathbf{L}\zeta_1 = \zeta_0.$$

As in the previous example, center manifold Theorem 4.18, gives

$$u = u_0 + \mathbf{\Psi}(u_0, \mu), \quad u_0 \in \mathcal{E}_0, \quad \mathbf{\Psi}(u_0, \mu) \in \mathcal{Z}_h,$$

and applying normal form Theorem 5.21 to the reduced system we find

$$u_0 = v_0 + \mathbf{\Phi}_\mu(v_0),$$

which gives the equality (5.43),

$$u \;=\; v_0 + \widetilde{\mathbf{\Psi}}(v_0, \mu), \quad v_0 \in \mathcal{E}_0, \quad \widetilde{\mathbf{\Psi}}(u_0, \mu) \in \mathcal{Z}.$$

For $v_0(t) \in \mathcal{E}_0$, we now write

$$v_0(t) \;=\; A(t)\zeta_0 + B(t)\zeta_1,$$

in which A and B are real-valued. According to the result in Lemma 5.9 and Remark 5.10, we find here the normal form

$$\begin{aligned} \frac{dA}{dt} &= B \\ \frac{dB}{dt} &= BP(A, \mu) + Q(A, \mu) + \rho(A, B, \mu), \end{aligned} \tag{5.61}$$

where $P(\cdot, \mu)$ and $Q(\cdot, \mu)$ are polynomials of degree p such that

$$P(0,0) = Q(0,0) = \frac{\partial Q}{\partial A}(0,0) = 0 \tag{5.62}$$

and

$$\rho(A, B, \mu) = o((|A| + |B|)^p).$$

Computation of the Normal Form We compute now the leading order terms in the expansion of the vector field. We set

$$\begin{aligned} \frac{dA}{dt} &= B \\ \frac{dB}{dt} &= \alpha_1(\mu) + \alpha_2(\mu)A + \alpha_3(\mu)B + \beta_1(\mu)AB + \beta_2(\mu)A^2 + \widetilde{\rho}(A, B, \mu), \end{aligned} \tag{5.63}$$

where the coefficients $\alpha_j(\mu)$ and $\beta_j(\mu)$ are such that

$$\alpha_j(\mu) = \alpha_j^{(1)}(\mu) + O(\mu^2), \quad \beta_j(\mu) = \beta_j^{(0)} + O(|\mu|),$$

with $\alpha_j^{(1)} : \mathbb{R}^m \to \mathbb{R}$ linear maps, according to (5.62), and $\widetilde{\rho}(A, B, \mu) = O(|A^2B| + |A|^3) + o((|A| + |B|)^p)$.

We proceed as for the previous example and start from identity (5.44), in which we replace the Taylor expansions of $\mathbf{R}$ and $\widetilde{\mathbf{\Psi}}$. With the notations from Section 5.2.3, we set here

$$\widetilde{\mathbf{\Psi}}(v_0, \mu) \;=\; \sum_{q+l+r \geq 1} A^q B^l \mathbf{\Psi}_{qlr}(\mu^{(r)}), \quad \mathbf{\Psi}_{100} = \mathbf{\Psi}_{010} = 0,$$

where $\mathbf{\Psi}_{ql0} \in \mathcal{Z}$, and $\mathbf{\Psi}_{qlr}$, $r \geq 1$, is r-linear symmetric in $\mu \in \mathbb{R}^m$ with values in $\mathcal{Z}$. By identifying in (5.44) the terms of order $O(A^2)$, $O(AB)$, and $O(B^2)$, we obtain

$$\beta_2^{(0)}\zeta_1 = \mathbf{L\Psi}_{200} + \mathbf{R}_{20}(\zeta_0, \zeta_0), \tag{5.64}$$

$$\beta_1^{(0)}\zeta_1 + 2\mathbf{\Psi}_{200} = \mathbf{L\Psi}_{110} + 2\mathbf{R}_{20}(\zeta_0, \zeta_1), \tag{5.65}$$

$$\mathbf{\Psi}_{110} = \mathbf{L\Psi}_{020} + \mathbf{R}_{20}(\zeta_1, \zeta_1), \tag{5.66}$$

and similarly, for the terms of order $O(\mu)$, $O(\mu A)$, and $O(\mu B)$, we find

$$\alpha_1^{(1)}\zeta_1 = \mathbf{L\Psi}_{001} + \mathbf{R}_{01}, \tag{5.67}$$

$$\alpha_2^{(1)}\zeta_1 + \alpha_1^{(1)}\mathbf{\Psi}_{110} = \mathbf{L\Psi}_{101} + \mathbf{R}_{11}(\zeta_0, \cdot) + 2\mathbf{R}_{20}(\zeta_0, \mathbf{\Psi}_{001}), \tag{5.68}$$

$$\alpha_3^{(1)}\zeta_1 + 2\alpha_1^{(1)}\mathbf{\Psi}_{020} + \mathbf{\Psi}_{101} = \mathbf{L\Psi}_{011} + \mathbf{R}_{11}(\zeta_1, \cdot) + 2\mathbf{R}_{20}(\zeta_1, \mathbf{\Psi}_{001}). \tag{5.69}$$

Notice that each term in these three equalities is a linear map of $\mu \in \mathbb{R}^m$ with values in $\mathcal{X}$, so that the equalities hold in $\mathcal{X}$ for any $\mu \in \mathbb{R}^m$.

Next, we claim that for an equation of the form

$$\mathbf{L\Psi} = \mathbf{R}, \tag{5.70}$$

with $\mathbf{R} \in \mathcal{X}$ and $\mathbf{\Psi} \in \mathcal{Z}$, a Fredholm alternative applies, just as in the previous example. Indeed, projecting again with the spectral projections $\mathbf{P}_0$ and $\mathbf{P}_h$, the equation decomposes as

$$\mathbf{L}_0\mathbf{P}_0\mathbf{\Psi} = \mathbf{P}_0\mathbf{R},$$
$$\mathbf{L}_h\mathbf{P}_h\mathbf{\Psi} = \mathbf{P}_h\mathbf{R}.$$

Since $\mathbf{L}_h : \mathcal{X}_h \to \mathcal{Z}_h$ is invertible, the second equation has the unique solution

$$\mathbf{P}_h\mathbf{\Psi} = \mathbf{L}_h{}^{-1}\mathbf{P}_h\mathbf{R}, \quad \mathbf{L}_h{}^{-1} : \mathcal{Z}_h \to \mathcal{X}_h.$$

The first equation is two-dimensional, and the linear operator $\mathbf{L}_0$ has a one-dimensional kernel spanned by ζ_0 and a two-dimensional generalized kernel spanned by ζ_0 and ζ_1. Then we can choose a dual basis $\{\zeta_{00}^*, \zeta_{01}^*\}$ for the generalized kernel of the adjoint $\mathbf{L}_0^*$, with the properties

$$\mathbf{L}_0^*\zeta_{01}^* = 0, \quad \mathbf{L}_0^*\zeta_{00}^* = \zeta_{01}^*,$$

and

$$\langle\zeta_0, \zeta_{00}^*\rangle = 1, \quad \langle\zeta_1, \zeta_{00}^*\rangle = 0, \quad \langle\zeta_0, \zeta_{01}^*\rangle = 0, \quad \langle\zeta_1, \zeta_{01}^*\rangle = 1.$$

The solvability condition is now

$$\langle\mathbf{P}_0\mathbf{R}, \zeta_{01}^*\rangle = 0,$$

and a solution $\mathbf{P}_0\mathbf{\Psi}$ is determined up an element in the kernel of $\mathbf{L}_0$. Among these solutions there is precisely one solution, $\mathbf{P}_0\widetilde{\mathbf{\Psi}}$, which is orthogonal to ζ_{00}^*, and summarizing we have that the solutions are of the form

$$\mathbf{P}_0\mathbf{\Psi} = \mathbf{P}_0\widetilde{\mathbf{\Psi}} + \alpha\zeta_0, \quad \langle\mathbf{P}_0\widetilde{\mathbf{\Psi}}, \zeta_{00}^*\rangle = 0, \quad \alpha \in \mathbb{R}.$$

Taking now the adjoint $\mathbf{P}_0^*$ of $\mathbf{P}_0$, we can rewrite the solvability condition

$$\langle \mathbf{R}, \zeta_1^* \rangle = 0, \quad \zeta_1^* = \mathbf{P}_0^* \zeta_{01}^*,$$

and the solutions

$$\mathbf{P}_0 \mathbf{\Psi} = \mathbf{P}_0 \widetilde{\mathbf{\Psi}} + \alpha \zeta_0, \quad \langle \widetilde{\mathbf{\Psi}}, \zeta_0^* \rangle = 0, \quad \zeta_0^* = \mathbf{P}_0^* \zeta_{00}^*, \quad \alpha \in \mathbb{R},$$

with $\widetilde{\mathbf{\Psi}}$ uniquely determined by the condition $\langle \widetilde{\mathbf{\Psi}}, \zeta_0^* \rangle = 0$, and α an arbitrary real number. In the case when the operator $\mathbf{L}$ has an adjoint $\mathbf{L}^*$ in $\mathcal{X}^*$, then ζ_0^* and ζ_1^* above are the vectors in the dual basis of the generalized kernel of the $\mathbf{L}^*$, with the properties

$$\mathbf{L}^* \zeta_1^* = 0, \quad \mathbf{L}^* \zeta_0^* = \zeta_1^*,$$

and

$$\langle \zeta_0, \zeta_0^* \rangle = 1, \quad \langle \zeta_1, \zeta_0^* \rangle = 0, \quad \langle \zeta_0, \zeta_1^* \rangle = 0, \quad \langle \zeta_1, \zeta_1^* \rangle = 1.$$

Notice that in this case again we have that the range of $\mathbf{L}$ is the space orthogonal to ζ_1^*.

Going back to the equalities (5.64)–(5.69), we can now determine the different coefficients in (5.63) from the solvability conditions, which give,

$$\begin{aligned}
\beta_2^{(0)} &= \langle \mathbf{R}_{20}(\zeta_0, \zeta_0), \zeta_1^* \rangle, \\
\beta_1^{(0)} &= \langle 2\mathbf{R}_{20}(\zeta_0, \zeta_1) - 2\mathbf{\Psi}_{200}, \zeta_1^* \rangle, \\
\alpha_1^{(1)}(\mu) &= \langle \mathbf{R}_{01}(\mu), \zeta_1^* \rangle, \\
\alpha_2^{(1)}(\mu) &= \langle -\alpha_1^{(1)}(\mu)\mathbf{\Psi}_{110} + \mathbf{R}_{11}(\zeta_0, \mu) + 2\mathbf{R}_{20}(\zeta_0, \mathbf{\Psi}_{001}(\mu)), \zeta_1^* \rangle, \\
\alpha_3^{(1)}(\mu) &= \langle -2\alpha_1^{(1)}(\mu)\mathbf{\Psi}_{020} - \mathbf{\Psi}_{101}(\mu) + \mathbf{R}_{11}(\zeta_1, \mu) + 2\mathbf{R}_{20}(\zeta_1, \mathbf{\Psi}_{001}(\mu)), \zeta_1^* \rangle.
\end{aligned}$$

Here, the terms $\mathbf{\Psi}_{200}$, $\mathbf{\Psi}_{110}$, $\mathbf{\Psi}_{001}(\mu)$, $\mathbf{\Psi}_{020}$ and $\mathbf{\Psi}_{101}(\mu)$ are obtained by solving successively the equations (5.64), (5.65), (5.67), (5.66), and (5.68), using the procedure explained above. First, from (5.64) we find

$$\mathbf{\Psi}_{200} = \widetilde{\mathbf{\Psi}}_{200} + \psi_{200}\zeta_0, \quad \langle \widetilde{\mathbf{\Psi}}_{200}, \zeta_0^* \rangle = 0, \quad \psi_{200} \in \mathbb{R},$$

and then from (5.65) we obtain

$$\mathbf{\Psi}_{110} = \widetilde{\mathbf{\Psi}}_{110} + 2\psi_{200}\zeta_1 + \psi_{110}\zeta_0, \quad \langle \widetilde{\mathbf{\Psi}}_{110}, \zeta_0^* \rangle = 0, \quad \psi_{110} \in \mathbb{R}.$$

The solvability condition for equation (5.66) determines the coefficient ψ_{200},

$$2\psi_{200} = \langle \mathbf{R}_{20}(\zeta_1, \zeta_1) - \widetilde{\mathbf{\Psi}}_{110}, \zeta_1^* \rangle,$$

and then solving (5.66) we find

$$\mathbf{\Psi}_{020} = \widetilde{\mathbf{\Psi}}_{020} + \psi_{110}\zeta_1 + \psi_{020}\zeta_0, \quad \langle \widetilde{\mathbf{\Psi}}_{020}, \zeta_0^* \rangle = 0, \quad \psi_{020} \in \mathbb{R}.$$

Next, from (5.67) we obtain

$$\mathbf{\Psi}_{001}(\mu) = \widetilde{\mathbf{\Psi}}_{001}(\mu) + \psi_{001}(\mu)\zeta_0, \quad \langle \widetilde{\mathbf{\Psi}}_{001}, \zeta_0^* \rangle = 0, \quad \psi_{001}(\mu) \in \mathbb{R},$$

and solving (5.68) we find

$$\begin{aligned}&\mathbf{\Psi}_{101}(\mu) = \widetilde{\mathbf{\Psi}}_{101}(\mu) + \alpha_1^{(1)}(\mu)\psi_{110}\zeta_1 + \psi_{101}(\mu)\zeta_0 + 2\psi_{001}(\mu)\widetilde{\mathbf{\Psi}}_{200},\\ &\langle \widetilde{\mathbf{\Psi}}_{101}, \zeta_0^*\rangle = 0, \quad \psi_{101}(\mu) \in \mathbb{R},\end{aligned}$$

where we have used in (5.68) the equality (5.64) which gives

$$2\mathbf{R}_{20}(\zeta_0, \mathbf{\Psi}_{001}(\mu)) = 2\mathbf{R}_{20}(\zeta_0, \widetilde{\mathbf{\Psi}}_{001}(\mu)) + 2\psi_{001}(\mu)(\beta_2^{(0)}\zeta_1 - \mathbf{L}\widetilde{\mathbf{\Psi}}_{200}).$$

Notice that we do not need to solve (5.69) and determine $\mathbf{\Psi}_{011}(\mu)$.

In the formulas above we have determined $\mathbf{\Psi}_{110}$, $\mathbf{\Psi}_{020}$, $\mathbf{\Psi}_{001}(\mu)$, and $\mathbf{\Psi}_{101}(\mu)$, up to an element $\psi_{110}\zeta_0$, $\psi_{020}\zeta_0$, $\psi_{001}(\mu)\zeta_0$, and $\psi_{101}(\mu)\zeta_0$, respectively, which belongs to the kernel of $\mathbf{L}$ and is arbitrary. The simplest choice is to take

$$\psi_{110} = \psi_{020} = \psi_{001}(\mu) = \psi_{101}(\mu) = 0.$$

However, notice that the coefficients $\beta_2^{(0)}$, $\beta_1^{(0)}$, and $\alpha_1^{(1)}(\mu)$ are uniquely determined, whereas $\alpha_2^{(1)}(\mu)$ and $\alpha_3^{(1)}(\mu)$ depend upon the choice of ψ_{110} and $\psi_{001}(\mu)$. We can then make use of the fact that ψ_{110} and $\psi_{001}(\mu)$ are arbitrary, in order to further simplify the normal form.

Further Transformation Consider the coefficient $\alpha_2^{(1)}(\mu)$ that we rewrite as

$$\begin{aligned}\alpha_2^{(1)}(\mu) \;=\; &\langle -\alpha_1^{(1)}(\mu)\widetilde{\mathbf{\Psi}}_{110} - 2\psi_{200}\mathbf{R}_{01}(\mu) + \mathbf{R}_{11}(\zeta_0, \mu) + 2\mathbf{R}_{20}(\zeta_0, \widetilde{\mathbf{\Psi}}_{001}(\mu)), \zeta_1^*\rangle\\ &+2\beta_2^{(0)}\psi_{001}(\mu).\end{aligned}$$

If the coefficient $\beta_2^{(0)} = 0$, then $\alpha_2^{(1)}(\mu)$ is uniquely determined. If $\beta_2^{(0)} \neq 0$, then we can choose the arbitrary coefficient $\psi_{001}(\mu)$ such that $\alpha_2^{(1)}(\mu) = 0$. Indeed, this is achieved by taking

$$\begin{aligned}\psi_{001}(\mu) \;=\; &\frac{1}{2\beta_2^{(0)}}\langle \alpha_1^{(1)}(\mu)\widetilde{\mathbf{\Psi}}_{110} + 2\psi_{200}\mathbf{R}_{01}(\mu)\\ &\qquad - \mathbf{R}_{11}(\zeta_0, \mu) - 2\mathbf{R}_{20}(\zeta_0, \widetilde{\mathbf{\Psi}}_{001}(\mu)), \zeta_1^*\rangle.\end{aligned}$$

Similarly, for $\alpha_3^{(1)}(\mu)$ we write

$$\begin{aligned}\alpha_3^{(1)}(\mu) \;=\; &\langle -2\alpha_1^{(1)}(\mu)\widetilde{\mathbf{\Psi}}_{020} - \widetilde{\mathbf{\Psi}}_{101}(\mu) + \mathbf{R}_{11}(\zeta_1, \mu) + 2\mathbf{R}_{20}(\zeta_1, \widetilde{\mathbf{\Psi}}_{001}(\mu)), \zeta_1^*\rangle\\ &-3\alpha_1^{(1)}(\mu)\psi_{110} + \beta_1^{(0)}\psi_{001}(\mu).\end{aligned}$$

Then if $\beta_1^{(0)} \neq 0$ we can take $\psi_{110} = 0$ and

$$\psi_{001}(\mu) = \frac{1}{\beta_1^{(0)}}\langle 2\alpha_1^{(1)}(\mu)\widetilde{\mathbf{\Psi}}_{020} + \widetilde{\mathbf{\Psi}}_{101}(\mu) - \mathbf{R}_{11}(\zeta_1, \mu) - 2\mathbf{R}_{20}(\zeta_1, \widetilde{\mathbf{\Psi}}_{001}(\mu)), \zeta_1^*\rangle,$$

and then $\alpha_3^{(1)}(\mu) = 0$.

Remark 5.32. *(i) Alternatively, we can obtain that either $\alpha_2^{(1)}(\mu) = 0$ or $\alpha_3^{(1)}(\mu) = 0$ by making a change of variables of the form*

$$\widetilde{A} = A - A_*(\mu),$$

with $A_*(\mu)$ suitably chosen, provided $\beta_2^{(0)} \neq 0$ or $\beta_1^{(0)} \neq 0$, respectively. Indeed, we have that $\alpha_2^{(1)}(\mu) = 0$ after the change of variables above, provided $A_*(\mu)$ satisfies

$$\frac{\partial Q(A_*(\mu), \mu)}{\partial A} = 0.$$

The existence of $A_*(\mu)$ with this property is obtained by solving equation

$$\frac{\partial Q(A, \mu)}{\partial A} = 0.$$

Since $\partial Q/\partial A(0,0) = 0$ and $\partial^2 Q/\partial A^2(0,0) = 2\beta_2^{(0)}$, the implicit function theorem gives a unique solution $A_*(\mu)$ of this equation for μ sufficiently small, provided $\beta_2^{(0)} \neq 0$. In a similar way, by solving $P(A_*(\mu), \mu) = 0$, for which $\partial P/\partial A(0,0) = \beta_1^{(0)}$, one finds $\alpha_3^{(1)}(\mu) = 0$ when $\beta_1^{(0)} \neq 0$.

(ii) An example of a second order ODE which has a normal form as described here is given in Section 5.2.4.

Reduced Dynamics The dynamics of systems of the form (5.63) have been extensively studied in the literature. In particular, we refer the reader to [20] for an analysis of the Takens–Bogdanov bifurcation, which is generically of codimension 2, arising for two small parameters, the coefficients α_1 and α_3 in (5.63). Varying these two coefficients, one finds here saddle-node, Hopf, and homoclinic bifurcations.

5.4.6 Example 5: $(i\omega_1)(i\omega_2)$ bifurcation

Consider again an equation of the form (5.41), with a parameter $\mu \in \mathbb{R}^m$ and satisfying the hypotheses in center manifold Theorem 4.18. We assume now that the spectrum of the linear operator $\mathbf{L}$ contains precisely two pairs of eigenvalues on the imaginary axis, $\pm i\omega_1$ and $\pm i\omega_2$, with $0 < \omega_1 < \omega_2$. Furthermore, we assume that these eigenvalues are simple, and that $\omega_1/\omega_2 = r/s \in \mathbb{Q}$, where r and s are positive integers, $r < s$, and the fraction is irreducible.

We point out that, since we can use as many parameters as needed, in practical situations when ω_1/ω_2 is irrational, or rational $\omega_1/\omega_2 = r/s$ with large r and s, then it is more convenient to consider these cases as perturbations of the case with $\omega_1/\omega_2 = r/s$, where r/s is a rational number, with smallest r and s, sufficiently close to ω_1/ω_2 (see also Remark 5.34).

Normal Form With these assumptions we have $\sigma_0 = \{\pm i\omega_1, \pm i\omega_2\}$, and the associated spectral subspace $\mathcal{E}_0$ is four-dimensional. We choose a basis $\{\zeta_1, \zeta_2, \overline{\zeta}_1, \overline{\zeta}_2\}$ in $\mathcal{E}_0$ consisting of the eigenvectors associated to the eigenvalues $i\omega_1$, $i\omega_2$, $-i\omega_1$, and $-i\omega_2$, respectively. As in the previous examples, center manifold Theorem 4.18, gives

$$u = u_0 + \mathbf{\Psi}(u_0,\mu), \quad u_0 \in \mathcal{E}_0, \quad \mathbf{\Psi}(u_0,\mu) \in \mathcal{Z}_h,$$

and applying normal form Theorem 5.21 to the reduced system we find

$$u_0 = v_0 + \mathbf{\Phi}_\mu(v_0),$$

which gives the equality (5.43),

$$u \;=\; v_0 + \widetilde{\mathbf{\Psi}}(v_0,\mu), \quad v_0 \in \mathcal{E}_0, \quad \widetilde{\mathbf{\Psi}}(u_0,\mu) \in \mathcal{Z}.$$

For $v_0(t) \in \mathcal{E}_0$, we now write

$$v_0(t) \;=\; A(t)\zeta_1 + B(t)\zeta_2 + \overline{A(t)\zeta_1} + \overline{B(t)\zeta_2},$$

in which A and B are complex-valued. According to the result in Lemma 5.15, we find here the normal form

$$\begin{aligned}
\frac{dA}{dt} &= i\omega_1 A + AP_1(|A|^2,|B|^2,A^s\overline{B}^r,\mu) + \overline{A}^{s-1}B^rP_2(|A|^2,|B|^2,\overline{A}^sB^r,\mu)\\
&\quad +\rho_1(A,B,\overline{A},\overline{B},\mu)\\
\frac{dB}{dt} &= i\omega_2 B + BQ_1(|A|^2,|B|^2,\overline{A}^sB^r,\mu) + A^s\overline{B}^{r-1}Q_2(|A|^2,|B|^2,A^s\overline{B}^r,\mu)\\
&\quad +\rho_2(A,B,\overline{A},\overline{B},\mu),
\end{aligned} \tag{5.71}$$

with P_j and Q_j polynomials in their first three arguments satisfying $P_1(0,0,0,0) = Q_1(0,0,0,0) = 0$, and $\rho_j(A,B,\overline{A},\overline{B},\mu) = O(|A|+|B|)^{2p+2})$, $j = 1,2$.

Computation of the Normal Form We proceed now as in the previous examples and compute the leading order terms in this normal form. We write

$$\begin{aligned}
\frac{dA}{dt} &= (i\omega_1 + \alpha_1(\mu))A + A(a|A|^2 + b|B|^2) + \beta_1\overline{A}^{s-1}B^r + \widetilde{\rho}_1(A,B,\overline{A},\overline{B},\mu)\\
\frac{dB}{dt} &= (i\omega_2 + \alpha_2(\mu))B + B(c|A|^2 + d|B|^2) + \beta_2 A^s\overline{B}^{r-1} + \widetilde{\rho}_2(A,B,\overline{A},\overline{B},\mu),
\end{aligned} \tag{5.72}$$

where

$$\alpha_j(\mu) = \alpha_j^{(1)}(\mu) + O(|\mu|^2), \quad j = 1,2,$$

with $\alpha_j^{(1)}$, $j = 1,2$, linear maps in μ, the coefficients a, b, c, d, β_1, and β_2 complex numbers, and

$$\widetilde{\rho}_j(A,B,\overline{A},\overline{B},\mu) = O(|\mu|(|A|+|B|)^3 + (|A|+|B|)^4 + |\mu|(|A|+|B|)^{r+s-1}).$$

Here $r+s \geq 3$, so that the coefficients β_1 and β_2 are relevant in this expansion only in the cases $(r,s) = (1,2)$ and $(r,s) = (1,3)$, which correspond to $\omega_2 = 2\omega_1$ and $\omega_2 = 3\omega_1$, respectively. Therefore the cases $(r,s) = (1,2)$ and $(r,s) = (1,3)$ are also called *strongly resonant cases*, whereas the cases when $r+s \geq 5$ are called *weakly resonant cases.*

The computation of these coefficients can be done exactly as in the previous two examples. We shall therefore only give the results here. First, by looking at the terms of orders $O(\mu A)$ and $O(\mu B)$ we obtain

$$\begin{aligned}
\alpha_1^{(1)} &= \langle \mathbf{R}_{11}(\zeta_1) + 2\mathbf{R}_{20}(\zeta_1, \mathbf{\Psi}_{00001}), \zeta_1^* \rangle, \\
\alpha_2^{(1)} &= \langle \mathbf{R}_{11}(\zeta_2) + 2\mathbf{R}_{20}(\zeta_2, \mathbf{\Psi}_{00001}), \zeta_2^* \rangle,
\end{aligned}$$

where

$$\mathbf{\Psi}_{00001} = -\mathbf{L}^{-1}\mathbf{R}_{01}.$$

Here ζ_1^* and ζ_2^* belong to $\mathcal{X}^*$, and span the orthogonal to the range of $i\omega_1 - \mathbf{L}$ and $i\omega_2 - \mathbf{L}$, respectively, just as the vector ζ^* constructed in Section 5.4.2. Next, by considering the terms of order 2 in $(A, \overline{A}, B, \overline{B})$, in the case $\omega_2 \neq 2\omega_1$, we find

$$\begin{aligned}
\mathbf{\Psi}_{20000} &= (2i\omega_1 - \mathbf{L})^{-1}\mathbf{R}_{20}(\zeta_1, \zeta_1), \\
\mathbf{\Psi}_{10100} &= 2(i(\omega_1 + \omega_2) - \mathbf{L})^{-1}\mathbf{R}_{20}(\zeta_1, \zeta_2), \\
\mathbf{\Psi}_{10010} &= 2(i(\omega_1 - \omega_2) - \mathbf{L})^{-1}\mathbf{R}_{20}(\zeta_1, \overline{\zeta_2}), \\
\mathbf{\Psi}_{11000} &= -2\mathbf{L}^{-1}\mathbf{R}_{20}(\zeta_1, \overline{\zeta_1}), \\
\mathbf{\Psi}_{00200} &= (2i\omega_2 - \mathbf{L})^{-1}\mathbf{R}_{20}(\zeta_2, \zeta_2), \\
\mathbf{\Psi}_{00110} &= -2\mathbf{L}^{-1}\mathbf{R}_{20}(\zeta_2, \overline{\zeta_2}),
\end{aligned}$$

whereas if $\omega_2 = 2\omega_1$ we need to solve the equations

$$\begin{aligned}
\beta_1\zeta_1 + (i\omega_1 - \mathbf{L})\overline{\mathbf{\Psi}_{10010}} &= 2\mathbf{R}_{2,0}(\zeta_2, \overline{\zeta_1}), \\
\beta_2\zeta_2 + (i\omega_2 - \mathbf{L})\mathbf{\Psi}_{20000} &= \mathbf{R}_{20}(\zeta_1, \zeta_1).
\end{aligned}$$

The solvability conditions for these two equations give the formulas for the coefficients β_1 and β_2, in this case,

$$\begin{aligned}
\beta_1 &= \langle 2\mathbf{R}_{20}(\zeta_2, \overline{\zeta_1}), \zeta_1^* \rangle, \\
\beta_2 &= \langle \mathbf{R}_{20}(\zeta_1, \zeta_1), \zeta_2^* \rangle.
\end{aligned}$$

Finally, by considering the terms of order 3, we find in the case $\omega_2 \neq 3\omega_1$ that

$$\begin{aligned}
a &= \langle 2\mathbf{R}_{20}(\zeta_1, \mathbf{\Psi}_{11000}) + 2\mathbf{R}_{20}(\overline{\zeta_1}, \mathbf{\Psi}_{20000}) + 3\mathbf{R}_{30}(\zeta_1, \zeta_1, \overline{\zeta_1}), \zeta_1^* \rangle, \\
b &= \langle 2\mathbf{R}_{20}(\zeta_1, \mathbf{\Psi}_{00110}) + 2\mathbf{R}_{20}(\zeta_2, \mathbf{\Psi}_{10010}) + 2\mathbf{R}_{20}(\overline{\zeta_2}, \mathbf{\Psi}_{10100}) \\
&\quad +6\mathbf{R}_{30}(\zeta_1, \zeta_2, \overline{\zeta_2}), \zeta_1^* \rangle, \\
c &= \langle 2\mathbf{R}_{20}(\zeta_1, \overline{\mathbf{\Psi}_{10010}}) + 2\mathbf{R}_{20}(\zeta_2, \mathbf{\Psi}_{11000}) + 2\mathbf{R}_{20}(\overline{\zeta_1}, \mathbf{\Psi}_{10100}) \\
&\quad +6\mathbf{R}_{30}(\zeta_1, \zeta_2, \overline{\zeta_1}), \zeta_2^* \rangle, \\
d &= \langle 2\mathbf{R}_{20}(\zeta_2, \mathbf{\Psi}_{00110}) + 2\mathbf{R}_{20}(\overline{\zeta_2}, \mathbf{\Psi}_{00200}) + 3\mathbf{R}_{30}(\zeta_2, \zeta_2, \overline{\zeta_2}), \zeta_2^* \rangle.
\end{aligned}$$

Exercise 5.33. *Compute the coefficients a, b, c, d, β_1, and β_2 in the case $\omega_2 = 3\omega_1$.*

Reduced Dynamics Finding the full bifurcation diagram of a parameter-dependent dynamical system in high dimensions is beyond the scope of these notes. This is also the case for the system (5.71), which is four-dimensional. Instead, the analysis is often restricted to the questions of finding bounded orbits, such as equilibria, periodic orbits, invariant tori, homoclinic or heteroclinic orbits, and determining their stability properties. In particular, one way of treating the existence question is to first show the existence of some bounded orbit for the truncated normal form, obtained by removing the small remainder ρ, e.g., by removing $\widetilde{\rho}$ in (5.72), and then show the persistence of this orbit for the full system. For equilibria and periodic orbits the persistence question can be often solved by an adapted implicit function theorem, but this question is much more delicate for invariant tori, homoclinics, and heteroclinics, and may be wrong. We discuss this type of difficulty in more detail in Chapter 4 of [21] in the case of reversible systems.

We do not attempt to discuss here these issues for the system (5.71), for which we refer for instance to [20]. Instead, we only mention some basic facts for the generic situation in which all the coefficients in (5.72) are nonzero. For the μ-dependent coefficients $\alpha_1(\mu)$ and $\alpha_2(\mu)$, which are small, since $\alpha_1(0) = \alpha_2(0) = 0$, we write

$$\alpha_j(\mu) = \nu_j + i\chi_j, \quad j = 1, 2,$$

and assume that the small real parts ν_j are nonzero. A convenient way of studying system (5.72) is in polar coordinates, by setting

$$A = r_1 e^{i\theta_1}, \quad B = r_2 e^{i\theta_2}.$$

Restricting ourselves to the leading order system obtained by removing the terms $\widetilde{\rho}_j$, $j = 1, 2$, in (5.72), we find three equations which decouple:

$$\begin{aligned}
\frac{dr_1}{dt} &= \nu_1 r_1 + r_1(a_r r_1^2 + b_r r_2^2) + r_1^{s-1} r_2^r Re(\beta_1 e^{-i\Theta}) \\
\frac{dr_2}{dt} &= \nu_2 r_2 + r_2(c_r r_1^2 + d_r r_2^2) + r_1^s r_2^{r-1} Re(\beta_2 e^{i\Theta}) \\
\frac{d\Theta}{dt} &= \gamma + (sa_i - rc_i) r_1^2 + (sb_i - rd_i) r_2^2 + r_1^{s-2} r_2^{r-2} \Im(r_2^2 s\beta_1 e^{-i\Theta} - r_1^2 r\beta_2 e^{i\Theta}),
\end{aligned} \tag{5.73}$$

in which

$$\Theta = s\theta_1 - r\theta_2,$$

together with an equation for θ_1. Here $\gamma = s\chi_1 - r\chi_2$ is a *detuning parameter*, and the subscripts r and i indicate the real and the imaginary parts, respectively, of a complex number.

In particular, the equilibria (r_1, r_2, Θ) of the three equations which decouple depend upon the values of the coefficients a, b, c, d, β_1, and β_2, and upon the three small parameters ν_1, ν_2, and γ. These equilibria correspond to periodic solutions for the four-dimensional truncated system (5.72), because of the additional phase θ_1, and, provided they persist, also for the full system (5.71).

Looking at (5.73) we notice again the fundamental difference between the weakly resonant cases where $r + s \geq 5$, and the strongly resonant cases where $r + s \leq 4$. Indeed, in the weakly resonant cases the Θ-dependent terms in the equations for r_1 and r_2 are of an order higher than 3, so that these two equations decouple in the truncation at order 3. One can first solve these two equations, for which we are in the presence of a bifurcation of codimension 2, with two small parameters ν_1 and ν_2. We refer to [20] for a detailed analysis of this situation. However, when including the higher order terms, we observe that the two first equations give the equilibria r_1 and r_2 as functions of Θ, which are generically of size $O((|\nu_1| + |\nu_2|)^{1/2})$. Then the equation for Θ leads to a condition between the small parameters γ and ν_1, ν_2, represented in the three-dimensional parameter space by a "resonance tongue."

Remark 5.34. *The case when ω_1/ω_2 is irrational is similar to the weakly resonant cases discussed above and can be analyzed in the same way. However, we point out that this irrationality condition is physically hard to check, so that in practical situations it is more convenient to regard this situation as a perturbation of a weakly resonant case by considering the closest rational number r/s to ω_1/ω_2 which has the smallest sum $r+s$, and then taking a detuning parameter $\delta = s\omega_1 - r\omega_2$, which is added to the detuning γ in the system (5.73). This allows us to regard this situation as a small perturbation of the case $\omega_1/\omega_2 = r/s$. On the contrary, in the strongly resonant cases the terms in $r_1^{s-1}r_2^r$ and $r_1^s r_2^{r-1}$ are of order 2 or 3, i.e., they are larger or comparable to the cubic terms. This introduces a number of difficulties in the bifurcation study. We refer to [49], and the references therein, for a discussion of the case $\omega_1/\omega_2 = 1/2$.*

Remark 5.35 ($(i\omega)^2$ bifurcation (1:1 resonance))**.** *In the same context as above, one can consider the case $\omega_1 = \omega_2$. The most interesting situation arises when these eigenvalues are double, non-semisimple. In this case the center manifold is four-dimensional and the normal form is given by Lemma 5.17. We refer to [16] for an analysis of the generic cases, in which one finds a bifurcation of codimension 3, i.e., involving three small parameters. In Chapter 4 of [21], we discuss this situation in the case of reversible systems, where it turns out that the bifurcation is of codimension 1, only.*

5.5 Further Normal Forms

5.5.1 Time-Periodic Normal Forms

A situation which arises quite often in applications is that of a periodically forced system. Here we consider the cases where the system is nonautonomous, as in Section 4.3.2, with $\mathbf{R}$ being periodic in t. In particular, this means that the time-dependency occurs as a small perturbation near the origin. This is not the general case of systems with time-periodic coefficients, and also not the case of autonomous systems near a closed orbit, for which normal forms may be found for general cases in [25, 26].

We consider a differential equation in $\mathbb{R}^n$ of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u, \mu, t), \tag{5.74}$$

for which we assume that the following hypothesis holds.

Hypothesis 5.36. *Assume that $\mathbf{L}$ and $\mathbf{R}$ in (5.74) have the following properties:*

(i) $\mathbf{L}$ is a linear map in $\mathbb{R}^n$;

(ii) *for some $k \geq 2$ and $l \geq 1$, there exists a neighborhood $\mathcal{V}$ of the origin in $\mathbb{R}^n \times \mathbb{R}^m$ such that the map $t \mapsto \mathbf{R}(\cdot,\cdot,t)$ belongs to $H^l(\mathbb{R}, \mathcal{C}^k(\mathcal{V}, \mathbb{R}^n))$;*

(iii) $\mathbf{R}(0,0,t) = 0$ *and* $D_u\mathbf{R}(0,0,t) = 0$ *for all* $t \in \mathbb{R}$*;*

(iv) *there exists $\tau > 0$, such that*

$$\mathbf{R}(u,\mu,t+\tau) = \mathbf{R}(u,\mu,t) \text{ for all } t \in \mathbb{R}, (u,\mu) \in \mathcal{V}.$$

Notice that the time dependency is taken in the Sobolev space H^l with $l \geq 1$. This is to insure that we can multiply two such functions, since $H^1(\mathbb{R}/\tau\mathbb{Z})$ is an algebra. We could use continuous functions instead, but H^l is really useful when we are looking at infinite-dimensional problems.

Theorem 5.37 (Periodically forced normal form)**.** *Consider the system (5.74) and assume that Hypothesis 5.36 holds. Then for any positive integer $p \leq k$ there exist neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of 0 in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and a τ-periodic function $t \mapsto \mathbf{\Phi}(\cdot,\cdot,t)$, which belongs to $H^l(\mathbb{R}/\tau\mathbb{Z}, \mathcal{C}^k(\mathbb{R}^n \times \mathcal{V}_2, \mathbb{R}^n))$, with the following properties:*

(i) *$\mathbf{\Phi}$ is a polynomial of degree p in its first argument, and the coefficients of the monomials of degree q belong to $H^l(\mathbb{R}/\tau\mathbb{Z}, \mathcal{C}^{k-q}(\mathcal{V}_2, \mathbb{R}^n))$. Furthermore,*

$$\mathbf{\Phi}(0,0,t) = 0, \quad D_u\mathbf{\Phi}(0,0,t) = 0 \text{ for all } t \in \mathbb{R}.$$

(ii) *For $v \in \mathcal{V}_1$, the polynomial change of variable*

$$u = v + \mathbf{\Phi}(v,\mu,t),$$

transforms system (5.25) into the "normal form"

$$\frac{dv}{dt} = \mathbf{L}v + \mathbf{N}(v,\mu,t) + \boldsymbol{\rho}(v,\mu,t), \tag{5.75}$$

with the following properties:

(a) *The map $t \mapsto \mathbf{N}(\cdot,\cdot,t)$ is τ-periodic and satisfies*

$$\mathbf{N}(0,0,t) = 0, \quad D_v\mathbf{N}(0,0,t) = 0 \text{ for all } t \in \mathbb{R}.$$

Furthermore, $\mathbf{N}$ is a polynomial of degree p in its first argument and the coefficients of the monomials of degree q belong to $H^l(\mathbb{R}/\tau\mathbb{Z}, \mathcal{C}^{k-q}(\mathcal{V}_2, \mathbb{R}^n))$.

(b) *The equality*

$$e^{t\mathbf{L}^*}\mathbf{N}(e^{-t\mathbf{L}^*}v,\mu,t) = \mathbf{N}(v,\mu,0) \tag{5.76}$$

holds for all $(t,v) \in \mathbb{R} \times \mathbb{R}^n$ and $\mu \in \mathcal{V}_2$.

(c) *The map $\boldsymbol{\rho}$ belongs to $H^l(\mathbb{R}/\tau\mathbb{Z}, \mathcal{C}^k(\mathcal{V}_1 \times \mathcal{V}_2, \mathbb{R}^n))$ and*

$$\boldsymbol{\rho}(v,\mu,t) = o(\|v\|^p) \text{ for all } (t,v) \in \mathbb{R} \times \mathcal{V}_1,\ \mu \in \mathcal{V}_2.$$

A preliminary version of this theorem appeared in [13].

Remark 5.38. *As in Theorem 5.2 we can replace (5.76) by*

$$\frac{\partial \mathbf{N}(v,\mu,t)}{\partial t} = D_v\mathbf{N}(v,\mu,t)\mathbf{L}^*v - \mathbf{L}^*\mathbf{N}(v,\mu,t) \text{ for all } (t,v) \in \mathbb{R}\times\mathbb{R}^n,\ \mu \in \mathcal{V}_2. \tag{5.77}$$

Via Fourier analysis this equation is, for every Fourier mode, of the same form as (5.5).

Proof. Proceeding as in the proof of Theorems 5.2 in Section 5.1.1, we are lead to solve the equation

$$\frac{\partial \mathbf{\Phi}}{\partial t} + \mathcal{A}_{\mathbf{L}}\mathbf{\Phi} + \mathbf{N} = \mathbf{\Pi}_p\left(\mathbf{R}(\cdot + \mathbf{\Phi},\mu) - D_v\mathbf{\Phi}\cdot\mathbf{N}\right) \tag{5.78}$$

with respect to $(\mathbf{\Phi},\mathbf{N})$, which are unknown functions of (v,μ,t). This equation is the analogue for this situation as in the proof of Theorem 5.21, and the notations $\mathcal{A}_{\mathbf{L}}$ and $\mathbf{\Pi}_p$ below have the same meaning as in this proof (see also equality (5.30)).
We start by solving the equation at $\mu = 0$. Then, at each degree q in v, we have to solve a linear equation of the form

$$\frac{\partial \mathbf{\Phi}}{\partial t} + \mathcal{A}_{\mathbf{L}}\mathbf{\Phi} = \mathbf{Q} - \mathbf{N}, \tag{5.79}$$

in which $\mathbf{Q} \in H^l(\mathbb{R}/\tau\mathbb{Z},\mathcal{H}_q)$, where $\mathcal{H}_q$ is the space of homogeneous polynomials of degree q, as in Section 5.1.1. Taking the Fourier expansion with respect to t of (5.79), we find for the kth Fourier coefficient,

$$\left(\frac{2ik\pi}{\tau} + \mathcal{A}_{\mathbf{L}}\right)\mathbf{\Phi}^{(k)} = \mathbf{Q}^{(k)} - \mathbf{N}^{(k)}.$$

This equation is now solved using the scalar product introduced in Section 5.1.1. It follows that we may choose $\mathbf{N}^{(k)}$ as the orthogonal projection of $\mathbf{Q}^{(k)}$ on the kernel of the adjoint of $(2ik\pi/\tau + \mathcal{A}_{\mathbf{L}})$, which is $(-2ik\pi/\tau + \mathcal{A}_{\mathbf{L}^*})$, and $\mathbf{\Phi}^{(k)}$ orthogonal to the kernel of $(2ik\pi/\tau + \mathcal{A}_{\mathbf{L}})$. In fact, this is equivalent to considering the scalar product in $L^2(\mathbb{R}/\tau\mathbb{Z},\mathcal{H}_q)$ defined through

$$\langle \mathbf{\Phi},\mathbf{\Psi}\rangle_\tau = \frac{1}{\tau}\int_0^\tau \langle \mathbf{\Phi}(\cdot,t), \overline{\mathbf{\Psi}}(\cdot,t)\rangle dt, \tag{5.80}$$

and then directly solving (5.79) with the help of the formal adjoint $-\partial/\partial t + \mathcal{A}_{\mathbf{L}^*}$ of the linear operator $\partial/\partial t + \mathcal{A}_{\mathbf{L}}$ in $L^2(\mathbb{R}/\tau\mathbb{Z},\mathcal{H})$.
The Fourier analysis above shows that there is a unique solution $(\mathbf{\Phi},\mathbf{N})$ of (5.79) satisfying

$$\mathbf{\Phi} \in H^{l+1}(\mathbb{R}/\tau\mathbb{Z},\mathcal{H}_q), \quad \mathbf{\Phi} \in \left(\ker\left(\frac{\partial}{\partial t} + \mathcal{A}_{\mathbf{L}}\right)\right)^\perp,$$

and

$$\mathbf{N} \in H^{l}(\mathbb{R}/\tau\mathbb{Z},\mathcal{H}_q), \quad \mathbf{N} \in \ker\left(-\frac{\partial}{\partial t} + \mathcal{A}_{\mathbf{L}^*}\right),$$

for any $\mathbf{Q} \in H^l(\mathbb{R}/\tau\mathbb{Z},\mathcal{H}_q)$. Furthermore, the linear mapping $\mathbf{Q} \mapsto (\mathbf{\Phi},\mathbf{N})$ is bounded from $H^l(\mathbb{R}/\tau\mathbb{Z},\mathcal{H}_q)$ to $(H^l(\mathbb{R}/\tau\mathbb{Z},\mathcal{H}_q))^2$.
Finally, we solve the equation (5.78) for small μ. The proof is done in the same way as the proof of Theorem 5.21 and we omit the details here. ∎

Remark 5.39. *Consider an infinite-dimensional system, as in section 4, and assume that the center manifold theorems , Theorem 4.24 with periodic time-dependence, and Theorem 4.18 for perturbed vector fields apply (e.g., see the example of periodically forced Hopf bifurcation below). We then obtain a reduced finite-dimensional system in $\mathcal{E}_0$ which is of the form (5.74). Hence, we can apply Theorem 5.37 to this reduced system. For the computation of the normal form, we can make it directly on the infinite-dimensional system, as it is not necessary to split the computation into the computation of the center manifold and the computation of the normal form, just as in the computation made in Section 5.4.*

5.5.2 Example: Periodically Forced Hopf Bifurcation

Consider an infinite-dimensional system of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u, \mu, t), \tag{5.81}$$

where, with the notations from section 4,

$$\mathbf{L} \in \mathcal{L}(\mathcal{Z}, \mathcal{X}) \text{ and } \mathbf{R} \in H^l(\mathbb{R}, \mathcal{C}^k(\mathcal{V}, \mathcal{Y})),$$

for $k \geq 2$, $l \geq 1$, and $\mathcal{V}$ a neighborhood of the origin in $\mathcal{Z} \times \mathbb{R}^m$. We assume that $\mathbf{R}$ is τ-periodic in t,

$$\mathbf{R}(u, \mu, t + \tau) = \mathbf{R}(u, \mu, t) \text{ for all } (u, \mu) \in \mathcal{V},\ t \in \mathbb{R},$$

and

$$\mathbf{R}(0, 0, t) = 0, \quad D_u\mathbf{R}(0, 0, t) = 0.$$

We further assume that the hypotheses of Theorems 4.24 and 4.18 are satisfied, and that the Hypothesis 4.4 on $\mathbf{L}$ holds with $\sigma_0 = \{\pm i\omega\}$, in which $\pm i\omega$ are simple eigenvalues.

Normal Form Under the above hypotheses, we find a 2-dimensional reduced system to which we can apply Theorem 5.37. We choose an eigenvector ζ associated with the eigenvalue $i\omega$, so that $\{\zeta, \overline{\zeta}\}$ is a basis of $\mathcal{E}_0$. As in the previous examples, we then have

$$u(t) = v_0(t) + \widetilde{\mathbf{\Psi}}(v_0(t), \mu, t),$$

with

$$v_0(t) = A(t)\zeta + \overline{A(t)\zeta} \in \mathcal{E}_0,$$

and $\widetilde{\mathbf{\Psi}}(v_0, \mu, t) \in \mathcal{Z}$, for (v_0, μ) in a neighborhood of 0 in $\mathcal{E}_0 \times \mathbb{R}^m$. Furthermore, $\mathbf{\Psi}$ is τ-periodic in t, and

$$\widetilde{\mathbf{\Psi}}(0, 0, t) = 0, \quad D_{v_0}\widetilde{\mathbf{\Psi}}(0, 0, t) = 0.$$

The normal form of the reduced equation is

$$\frac{dA}{dt} = i\omega A + N(A, \overline{A}, \mu, t) + \rho(A, \overline{A}, \mu, t),$$

with N polynomial of degree p in $(A, \overline{A})$, with coefficients depending upon μ and t, as in Theorem 5.37, and

$$\rho(A, \overline{A}, \mu, t) = O(|A|^{p+1}).$$

Moreover,

$$N(0,0,0,t) = 0, \partial_A N(0,0,0,t) = \partial_{\overline{A}} N(0,0,0,t) = 0,$$
$$N(A,\overline{A},\mu,t+\tau) = N(A,\overline{A},\mu,t),$$

and the identity (5.76) gives in this case

$$e^{-i\omega t}N(e^{i\omega t}A, e^{-i\omega t}\overline{A},\mu,t) \quad = \quad N(A,\overline{A},\mu,0) \tag{5.82}$$

for all $A \in \mathbb{C}$ and $t \in \mathbb{R}$.

We set $\omega_f = 2\pi/\tau$, and consider the monomials of the nth Fourier mode of $N(A,\overline{A},\mu,\cdot)$ of the form

$$\alpha_{pq}^{(n)}(\mu)A^p\overline{A}^q e^{in\omega_f t}.$$

According to (5.82), these monomials should satisfy

$$\alpha_{pq}^{(n)}(\mu)\left(e^{i\left((p-q-1)\omega+n\omega_f\right)t} - 1\right) = 0,$$

so that

$$(p-q-1)\omega + n\omega_f = 0. \tag{5.83}$$

Assume now that

$$\frac{\omega_f}{\omega} = \frac{r}{s} \in \mathbb{Q}.$$

Then the equality (5.83) leads to

$$p-q-1 = lr, \quad n = -ls, \quad l \in \mathbb{Z},$$

and we conclude that

$$N(A,\overline{A},\mu,t) = AN_0(|A|^2,(Ae^{-i\omega t})^r,\mu) + \overline{A}^{r-1}e^{ri\omega t}N_1(|A|^2,(\overline{A}e^{i\omega t})^r,\mu), \tag{5.84}$$

where N_0 and N_1 are polynomials in their first two arguments.

The leading order terms in the normal form now strongly depend upon the value of r. For $r = 1$ we find the truncated equation

$$\begin{aligned}\frac{dA}{dt} &= i\omega A + a(\mu)A + c(\mu)e^{i\omega t} + d(\mu)\overline{A}e^{2i\omega t} + e(\mu)A^2e^{-i\omega t} + f(\mu)\overline{A}^2e^{3i\omega t} \\ &+ b(\mu)A|A|^2 + g(\mu)A^3e^{-2i\omega t} + h(\mu)\overline{A}^3e^{4i\omega t} + j(\mu)\overline{A}|A|^2e^{2i\omega t},\end{aligned} \tag{5.85}$$

where $a(0) = c(0) = d(0) = 0$. For $r = 2$ we obtain the equation

$$\begin{aligned}\frac{dA}{dt} &= i\omega A + a(\mu)A + c(\mu)\overline{A}e^{2i\omega t} + b(\mu)A|A|^2 \\ &+ d(\mu)A^3e^{-2i\omega t} + g(\mu)\overline{A}^3e^{4i\omega t} + f(\mu)\overline{A}|A|^2e^{2i\omega t},\end{aligned} \tag{5.86}$$

with $a(0) = c(0) = 0$, whereas for $r \geq 3$ we find

$$\frac{dA}{dt} = i\omega A + a(\mu)A + b(\mu)A|A|^2 + c(\mu)\overline{A}^{r-1}e^{ri\omega t}, \tag{5.87}$$

in which $a(0) = 0$. The cases $r = 1, 2, 3$ are strongly resonant cases, leading to very rich dynamics, the “worse” being $r = 1$.

Remark 5.40. *(i) In the case of a small periodic forcing, i.e., if*

$$\partial_t \mathbf{R}(u, 0, t) = 0,$$

all the coefficients of the time-dependent terms in the above equations vanish at $\mu = 0$. *We refer to [15] for an analysis of the dynamics in the cases* $r = 1$ *and* $r = 2$.

(ii) The case when ω_f/ω *is irrational is quite academic, since it is physically hard to check. Instead, it is more convenient to consider this case as a small perturbation of the case* $\omega_f/\omega = r/s \in \mathbb{Q}$, *by choosing a rational number* r/s *with minimal* r *close enough to* ω_f/ω.

Computation of the Normal Form We briefly describe below how to compute the terms of order $O(\mu)$ of the coefficients $a(\mu)$ and $c(\mu)$, and the coefficients $b(0)$, $d(0)$, $g(0)$, and $f(0)$ in the case $\omega_f = 2\omega$, i.e., $r = 2$ and $s = 1$.
We set

$$a(\mu) = a^{(1)}(\mu) + O(|\mu|^2), \quad c(\mu) = c^{(1)}(\mu) + O(|\mu|^2),$$

where $a^{(1)}$ and $c^{(1)}$ are linear maps in $\mu \in \mathbb{R}^m$. We proceed as in the previous examples by taking the Taylor expansions of $\mathbf{R}$ and $\mathbf{\Psi}$. With similar notations, we first find at order $O(\mu)$

$$\frac{d\mathbf{\Psi}_{001}}{dt} - \mathbf{L}\mathbf{\Psi}_{001} = \mathbf{R}_{01}(t).$$

Here $\mathbf{R}_{01}(t)$ is τ-periodic, and after taking its Fourier expansion

$$\mathbf{R}_{01}(t) = \sum_{n\in\mathbb{Z}} \mathbf{R}_{01}^{(n)} e^{in\omega_f t}, \quad \mathbf{R}_{01}^{(n)} \in \mathcal{L}(\mathbb{R}^m, \mathcal{Y}),$$

we then have to solve the equations

$$(in\omega_f - \mathbf{L})\, \mathbf{\Psi}_{001}^{(n)} = \mathbf{R}_{01}^{(n)}$$

for any $n \in \mathbb{Z}$. Since $\omega_f = 2\omega$, the operators $(in\omega_f - \mathbf{L})$ are invertible, so that we obtain a unique solution $\mathbf{\Psi}_{001} \in H^l(\mathbb{R}/\tau\mathbb{Z}, \mathcal{L}(\mathbb{R}^m, \mathcal{Z})$.
Next, we consider the terms of order $O(\mu A)$ and find

$$\frac{d\mathbf{\Psi}_{101}}{dt} + (i\omega - \mathbf{L})\mathbf{\Psi}_{101} + a_1\zeta + \overline{c}_1 e^{-2i\omega t}\overline{\zeta} = \mathbf{R}_{11}(\zeta)(t) + 2\mathbf{R}_{20}(\zeta, \mathbf{\Psi}_{001}(t))(t).$$

Using again Fourier series, we obtain a system of equations for $n \in \mathbb{Z}$ as above. These equations are invertible for $n \notin \{0, -1\}$ and the solvability conditions for $n = 0$ and $n = -1$ determine the coefficients

$$\begin{aligned} a^{(1)} &= \langle \mathbf{R}_{11}(\zeta)(\cdot) + 2\mathbf{R}_{20}(\zeta, \mathbf{\Psi}_{001}(\cdot))(\cdot), \zeta^* \rangle_\tau \\ \overline{c}^{(1)} &= \langle \mathbf{R}_{11}(\zeta)(\cdot) + 2\mathbf{R}_{20}(\zeta, \mathbf{\Psi}_{001}(\cdot))(\cdot), e^{2i\omega t}\overline{\zeta}^* \rangle_\tau. \end{aligned}$$

Here $\langle \cdot, \cdot \rangle_\tau$ is the scalar product defined through (5.80), and ζ^* is taken such that $\{\zeta^*, \overline{\zeta^*}\}$ is a dual basis of $\{\zeta, \overline{\zeta}\}$ in $\mathcal{E}_0$.

Finally, we compute the coefficients $b(0)$, $d(0)$, $g(0)$, and $f(0)$ by considering successively the terms of orders $O(A^2)$, $O(A\overline{A})$, $O(A^3)$, and $O(A^2\overline{A})$. At orders $O(A^2)$ and $O(A\overline{A})$, we find

$$\begin{aligned}\frac{d\mathbf{\Psi}_{200}}{dt} + (2i\omega - \mathbf{L})\mathbf{\Psi}_{200} &= \mathbf{R}_{20}(\zeta,\zeta)(t),\\ \frac{d\mathbf{\Psi}_{110}}{dt} - \mathbf{L}\mathbf{\Psi}_{110} &= 2\mathbf{R}_{20}(\zeta,\overline{\zeta})(t),\end{aligned}$$

and $\mathbf{\Psi}_{200}(t)$ and $\mathbf{\Psi}_{110}(t)$ are determined just as $\mathbf{\Psi}_{001}$ above. At orders $O(A^3)$, and $O(A^2\overline{A})$ we obtain

$$\begin{aligned}\frac{d\mathbf{\Psi}_{300}}{dt} &+ (3i\omega - \mathbf{L})\mathbf{\Psi}_{300} + de^{-2i\omega t}\zeta + \overline{g}e^{-4i\omega t}\overline{\zeta}\\ &= 2\mathbf{R}_{20}(\zeta,\mathbf{\Psi}_{200}(t))(t) + \mathbf{R}_{30}(\zeta,\zeta,\zeta)(t),\\ \frac{d\mathbf{\Psi}_{210}}{dt} &+ (i\omega - \mathbf{L})\mathbf{\Psi}_{210} + b\zeta + \overline{f}e^{-2i\omega t}\overline{\zeta}\\ &= 2\mathbf{R}_{20}(\overline{\zeta},\mathbf{\Psi}_{200}(t))(t) + 3\mathbf{R}_{30}(\zeta,\zeta,\overline{\zeta})(t) + 2\mathbf{R}_{20}(\zeta,\mathbf{\Psi}_{110}(t))(t),\end{aligned}$$

and the coefficients are obtained from the solvability conditions for these equations:

$$\begin{aligned}b(0) &= \langle 2\mathbf{R}_{20}(\overline{\zeta},\mathbf{\Psi}_{200}(\cdot))(\cdot) + 3\mathbf{R}_{30}(\zeta,\zeta,\overline{\zeta})(\cdot) + 2\mathbf{R}_{20}(\zeta,\mathbf{\Psi}_{110}(\cdot))(\cdot),\zeta^*\rangle_\tau,\\ \overline{f(0)} &= \langle 2\mathbf{R}_{20}(\overline{\zeta},\mathbf{\Psi}_{200}(\cdot))(\cdot) + 3\mathbf{R}_{30}(\zeta,\zeta,\overline{\zeta})(\cdot) + 2\mathbf{R}_{20}(\zeta,\mathbf{\Psi}_{110}(\cdot))(\cdot),e^{2i\omega t}\overline{\zeta}^*\rangle_\tau,\\ d(0) &= \langle 2\mathbf{R}_{20}(\zeta,\mathbf{\Psi}_{200}(\cdot))(\cdot) + \mathbf{R}_{30}(\zeta,\zeta,\zeta)(\cdot),e^{2i\omega t}\zeta^*\rangle_\tau,\\ \overline{g(0)} &= \langle 2\mathbf{R}_{20}(\zeta,\mathbf{\Psi}_{200}(\cdot))(\cdot) + \mathbf{R}_{30}(\zeta,\zeta,\zeta)(\cdot),e^{4i\omega t}\overline{\zeta}^*\rangle_\tau.\end{aligned}$$

Exercise 5.41 (Periodically forced vibrating structure)**.** *Consider a system in* $\mathbb{R}^n$, $n = 2m$, *of the form*

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u,t),$$

in which $\mathbf{L}$ *and* $\mathbf{R}$ *have the following properties:*

(i) the linear map $\mathbf{L}$ *has* $2m$ *simple, purely imaginary eigenvalues* $\pm i\omega_j$, $j = 1,2,\dots,m$;

(ii) the map $\mathbf{R}$ *is smooth and* τ*-periodic in* t;

(iii) $\mathbf{R}(0,t) = D_u\mathbf{R}(0,t) = 0$ *for all* $t \in \mathbb{R}$.

Further consider the change of variables in the normal form Theorem 5.37,

$$u = v + \mathbf{\Phi}(v,t),$$

with $\mathbf{\Phi}$ *polynomial in* v, τ*-periodic in* t, *satisfying* $\mathbf{\Phi}(0,t) = D_v\mathbf{\Phi}(0,t) = 0$, *and with*

$$v = \sum_{j=1}^m A_j\zeta_j + \sum_{j=1}^m \overline{A}_j\overline{\zeta}_j,$$

where ζ_j *are the eigenvectors associated with the eigenvalues* $i\omega_j$*.*
Set $\omega_f = 2\pi/\tau$*, and take* r_0 *and* r_j*,* $j = 1, \ldots, m$ *a set of integers, such that*

$$r_0\omega_f + \sum_{j=1}^{m} r_j\omega_j = 0, \quad r_0 \neq 0, \tag{5.88}$$

with a minimal total degree $|r|$ *defined by*

$$|r| = \sum_{j=1}^{m} |r_j|.$$

Assuming the nonresonance condition

$$\sum_{j=1}^{m} \alpha_j\omega_j \neq 0, \quad \alpha_j \in \mathbb{Z}, \quad |\alpha| \leq p+1$$

for some $p \geq 3$*, show that*

(i) the normal form at order p reads

$$\frac{dA_j}{dt} = i\omega_j A_j + A_j P_j(|A_1|^2 + \cdots + |A_m|^2) + Q_j(A_1, \ldots, A_m, \overline{A}_1, \ldots, \overline{A}_m, t),$$

where P_j *are polynomials, and* Q_j *are polynomials in* $(A_1, \ldots, A_m, \overline{A}_1, \ldots, \overline{A}_m)$ *with* τ*-periodic coefficients in* t*;*

(ii) the lowest order monomials in the normal form that have time-dependent coefficients are of degree $|r|-1$*, and their coefficients are proportional to either* $e^{ir_0\omega_f t}$ *or* $e^{-ir_0\omega_f t}$*.*

Application: *Take* $m = 3$ *and assume that the eigenvalues* $\pm i\omega_1, \pm i\omega_2, \pm i\omega_3$ *of* $\mathbf{L}$ *satisfy*

$$\sum_{j=1}^{3} \alpha_j\omega_j \neq 0,$$

for any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$ *with* $|\alpha| \leq 4$*. Further assume that the frequency* ω_f *of the periodic forcing satisfies*

$$4\omega_f + \omega_1 - 3\omega_2 = 0,$$

and that no other integer combination corresponding to the minimal degree $|r| = 4$ *exists.*

(i) Show that the normal form at order 3 contains only *the following time-dependent terms:* $c_1 e^{-4i\omega_f t} A_2^3$ *in the equation for* A_1*, and* $c_2 e^{4i\omega_f t} A_1 \overline{A}_2^2$ *in the equation for* A_2*, with complex coefficients* c_1 *and* c_2*.*

(ii) Consider polar coordinates $\theta_j = \arg A_j$ *and set* $\Theta = \theta_1 - 3\theta_2 + 4\omega_f t$*. Show that the normal form at order 3 written in these polar coordinates leads to a four-dimensional autonomous system for* (r_1, r_2, r_3, Θ)*, which decouples from the two equations for the phases* θ_1 *and* θ_3*.*

5.5.3 Normal Forms for Analytic Vector Fields

An interesting issue about normal forms arises when the vector field in (5.25) is analytic in (u, μ). The polynomials $\mathbf{\Phi}$ and $\mathbf{N}$ exist for any order $p \in \mathbb{N}$, and a natural question is then the convergence of the series resulting as $p \to \infty$. In general this series does not converge, but under suitable conditions, it is possible to determine an optimal degree for the normal form polynomial that minimizes the remainder term $\boldsymbol{\rho}$ (in the sense that the remainder is exponentially small). We present in this section two recent results by Iooss and Lombardi [31, 32] which show the existence of this optimal degree.

Definition 5.42. *Consider a linear map* $\mathbf{L}$ *on* $\mathbb{C}^n$ *with eigenvalues* $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. *Set* $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, *and consider* $\gamma > 0$ *and* $\tau > n - 1$. *The linear map* $\mathbf{L}$ *is called* (γ, τ)-homologically diophantine *if for every* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, *with* $|\alpha| \geq 2$, *where* $|\alpha| = \sum_{j=1}^{n} \alpha_j$, *the following inequality holds:*

$$|\langle \lambda, \alpha \rangle - \lambda_j| \geq \frac{\gamma}{|\alpha|^{\tau}},$$

whenever $\langle \lambda, \alpha \rangle - \lambda_j \neq 0$.

The following result is proved in [31].

Theorem 5.43 (Optimal normal form)**.** *Consider the system (5.25) with* $\mathbf{R}$ *an analytic map in a neighborhood of the origin in* $\mathbb{R}^n \times \mathbb{R}^m$ *such that* $\mathbf{R}(0, \mu) = 0$ *for all* μ. *Assume that there exist positive constants* c *and* r *such that in the expansion*

$$\mathbf{R}(u, \mu) = \sum_{k+l \geq 2, k \geq 1} \mathbf{R}_{kl}(u^{(k)}, \mu^{(l)})$$

of $\mathbf{R}$, *the* $(k + l)$*-linear maps* $\mathbf{R}_{kl}$ *on* $(\mathbb{R}^n)^k \times (\mathbb{R}^m)^l$ *satisfy*

$$\|\mathbf{R}_{kl}(u_1, \dots, u_k, \mu_1, \dots, \mu_l)\| \leq c \frac{\|u_1\| \dots \|u_k\| \, \|\mu_1\| \dots \|\mu_l\|}{r^{k+l}}.$$

Then for any $p \geq 2$, *the result in Theorem 5.21 holds, with* $\mathbf{\Phi}$ *and* $\mathbf{N}$ *polynomials of degree* p *in* (u, μ). *Furthermore, the following properties hold:*

(i) If the linear operator $\mathbf{L}$ *is diagonalizable and* (γ, τ)*-homologically diophantine, then there is a degree* p_{opt} *for the polynomials* $\mathbf{\Phi}$ *and* $\mathbf{N}$ *such that the remainder* $\boldsymbol{\rho}$ *satisfies*

$$\sup_{\|v\| + \|\mu\| \leq \delta} \|\boldsymbol{\rho}(v, \mu)\| \leq M(\tau) e^{-C/\delta^b},$$

where C *depends upon* (c, r, γ, n, m), $M(\tau)$ *depends upon* τ *and* (c, r, γ, n, m), $b = (1 + \tau)^{-1}$, *and* $p_{opt} = O(\delta^{-b})$.

(ii) If 0 *is the only eigenvalue of* $\mathbf{L}$, *with at most one* 2×2 *or* 3×3 *Jordan block, then the above estimate for* $\boldsymbol{\rho}$ *holds with* $b = 1$.

Remark 5.44. *(i) Notice that the optimal degree p_{opt} of the normal form depends upon the radius of the ball where the remainder $\boldsymbol{\rho}$ is estimated. In applications, this is not really a restriction, since one may choose δ to be of order $O(|\mu|^{\beta})$ with $\beta > 0$ small enough, such that the "interesting dynamics" take place in a smaller ball. In particular, the bifurcating solutions lie inside this ball. The result shows that, under the above hypotheses, the remainder $\boldsymbol{\rho}$ is exponentially small with respect to the relevant terms in the bifurcation analysis of the normal form.*

(ii) The restriction (i) on the linear map $\mathbf{L}$ in Theorem 5.43 may sometimes be overcome by using a suitable decomposition of the problem (see [30] where the case $0^{2+}i\omega$, with $\mathbf{L}$ not diagonalizable, is studied).

A key ingredient in Theorem 5.43 is the analyticity of $\mathbf{R}$. However, this condition is not satisfied by the reduced systems given by the center manifold theorem, of interest here, in which the vector field is not analytic, even when the original vector field is analytic (see Remark 4.13). In this situation, the idea is to first use a normal form transformation on a suitably decomposed system, taking advantage of the analyticity of the vector field, and then use the center manifold reduction, taking into account the exponentially small estimate given by the normal form. In this context, the following result has been proved in [32].

Theorem 5.45. *Consider the system (5.25) with $\mathbf{R}$ as in Theorem 5.43. Further assume that $\mathbf{L}$ is the direct sum of two linear maps $\mathbf{L}_0$ on $\mathbb{R}^{n_0}$ and $\mathbf{L}_1$ on $\mathbb{R}^{n_1}$, with $n_0 + n_1 = n$, such that $\mathbf{L}_0$ is diagonalizable with eigenvalues $\lambda_1^{(0)}, \ldots, \lambda_{n_0}^{(0)}$, and that there exist positive constants γ and τ such that*

$$|\langle \alpha, \lambda^{(0)} \rangle - \lambda_j^{(1)}| \geq \frac{\gamma}{|\alpha|^{\tau}}, \quad j = 1, \ldots, n_1, \tag{5.89}$$

for any $\alpha \in \mathbb{N}^{n_0}$, $\alpha \neq 0$, where $\lambda^{(0)} = (\lambda_1^{(0)}, \ldots, \lambda_{n_0}^{(0)})$ and $\lambda_1^{(1)}, \ldots, \lambda_{n_1}^{(1)}$ are the eigenvalues of $\mathbf{L}_1$. Then, there exists a polynomial $\boldsymbol{\Phi} : \mathbb{R}^{n_0} \times \mathbb{R}^m \to \mathbb{R}^{n_1}$ of optimal degree p_{opt} such that the change of variables

$$u_1 = \widetilde{u}_1 + \boldsymbol{\Phi}(u_0, \mu),$$

transforms the system (5.25) into the following system in $\mathbb{R}^{n_0} \times \mathbb{R}^{n_1}$

$$\begin{aligned} \frac{du_0}{dt} &= \mathbf{L}_0 u_0 + \widetilde{\mathbf{R}}_0(u_0, \widetilde{u}_1, \mu), \\ \frac{d\widetilde{u}_1}{dt} &= \mathbf{L}_1 \widetilde{u}_1 + \widetilde{\mathbf{R}}_1(u_0, \widetilde{u}_1, \mu) + \boldsymbol{\rho}_1(u_0, \mu), \end{aligned} \tag{5.90}$$

in which $\widetilde{\mathbf{R}}_0$, $\widetilde{\mathbf{R}}_1$, and $\boldsymbol{\rho}_1$ are analytic in their arguments,

$$\widetilde{\mathbf{R}}_0(u_0, \widetilde{u}_1, \mu) = \mathbf{P}_0 \mathbf{R}(u_0 + \widetilde{u}_1 + \boldsymbol{\Phi}(u_0, \mu), \mu),$$

where $\mathbf{P}_0$ is the projection on the subspace $\mathbb{R}^{n_0}$,

$$\widetilde{\mathbf{R}}_1(u_0, \widetilde{u}_1, \mu) = O(\|\widetilde{u}_1\|(\|u_0\| + \|\widetilde{u}_1\| + \|\mu\|)),$$

and, with the notations from Theorem 5.43, $p_{opt} = O(\delta^{-b})$ where $b = (1 + \nu\tau)^{-1}$, ν being the maximal algebraic multiplicity of eigenvalues of $\mathbf{L}_1$, and

$$\sup_{\|u_0\| + \|\mu\| \leq \delta} \|\boldsymbol{\rho}_1(u_0, \mu)\| \leq M(\tau) e^{-C/\delta^b}.$$

Remark 5.46. *The polynomial $\mathbf{\Phi}$ in the above theorem satisfies the identity*

$$\begin{aligned} D_{u_0}\mathbf{\Phi}(u_0,\mu)\mathbf{L}_0 u_0 - \mathbf{L}_1\mathbf{\Phi}(u_0,\mu) &= -D_{u_0}\mathbf{\Phi}(u_0,\mu)\mathbf{P}_0\mathbf{R}(u_0+\mathbf{\Phi}(u_0,\mu),\mu) \\ &\quad + \mathbf{P}_1\mathbf{R}(u_0+\mathbf{\Phi}(u_0,\mu),\mu) - \boldsymbol{\rho}_1(u_0,\mu). \end{aligned}$$

From this identity one can compute the coefficients of the polynomial $\mathbf{\Phi}$ by identifying the powers of (u_0,μ) in the Taylor expansions of both sides (like in the computation described in Section 5.4; see also Figure 5.1). The Theorem 5.45 asserts that there is an optimal degree for the polynomial $\mathbf{\Phi}$ for which the remainder $\boldsymbol{\rho}_1$ is exponentially small.

A particularly interesting situation arises when in Theorem 5.45 the spectrum of $\mathbf{L}_0$ lies on the imaginary axis, whereas the spectrum of $\mathbf{L}_1$ is hyperbolic, i.e., it has no point on the imaginary axis. In this case the condition (5.89) is always satisfied, so that the result in Theorem 5.45 holds. Notice that if $\boldsymbol{\rho}_1$ would be identically 0, then the manifold $\widetilde{u}_1 = 0$, i.e.,

$$\{u = u_0 + u_1 = u_0 + \mathbf{\Phi}(u_0,\mu)\ ;\ u_0 \in \mathbb{R}^{n_0}\},$$

would be an invariant center manifold for the system (5.25). This means that we have found in Theorem 5.45 an approximated center manifold, with an exponentially small error, but with the property of keeping the analyticity of the vector field. Applying now the center manifold Theorem 4.18 to the system (5.90) one finds a reduced system for $u_0 \in \mathbb{R}^{n_0}$ in which the vector field is the sum of an analytic vector field with an exponentially small remainder, and it is possible to adapt the Theorem 5.43 for this reduced system. This result can be generalized to the infinite dimensional situation treated in Section 4.3.1. More precisely, we have the following result.

Theorem 5.47. *Consider equation (5.41), under the hypotheses of the center manifold Theorem 4.18. With the notations from Section 4.3.1, further assume that $\mathbf{R}$ is analytic on $\mathcal{V}_u \times \mathcal{V}_\mu$ and that $\mathbf{L}_0$ is diagonalizable. Then, there exists a polynomial $\mathbf{\Phi} : \mathcal{E}_0 \times \mathbb{R}^m \to \mathcal{Z}_h$ of optimal degree p_{opt}, such that the change of variable*

$$u_h = \widetilde{u}_h + \mathbf{\Phi}(u_0,\mu)$$

transforms equation (5.41) into a system of the form (5.90) for $u_0 \in \mathcal{E}_0$ and $\widetilde{u}_h \in \mathcal{Z}_h$, with the same properties as in Theorem 5.45 where the subscript 1 is replaced by h.

Remark 5.48. *As in the finite-dimensional case, one can apply center manifold Theorem 4.18 to the system given by the theorem above, and find a center manifold of the form $\{u = u_0 + \mathbf{\Phi}(u_0,\mu) + O(e^{-C/\delta})\ ;\ u_0 \in \mathcal{E}_0\}$ in a ball of radius δ in $\mathcal{Z}$. Again, it is possible to adapt Theorem 5.43 for the reduced system.*

Remark 5.49. *Another interesting situation in Theorem 5.45 arises when the eigenvalues of $\mathbf{L}_0$ and $\mathbf{L}_1$ are all purely imaginary. Provided they satisfy the condition (5.89), the result of the theorem allows us to give a bound for the solutions of the initial value problem, for initial values lying on the manifold $\{u = u_0 + \mathbf{\Phi}(u_0,\mu); u_0 \in \mathcal{E}_0\}$. One expects that $\widetilde{u}_1$ stays exponentially close to 0 for a very long time, i.e., we don't see the eigenmodes of $\mathbf{L}_1$ for a very long time of order $O(\delta^{-(b+1/\nu)})$, where ν is the maximal index of the eigenvalues of $\mathbf{L}_1$ (see [32]). This situation occurs for instance in the theory of nonlinear vibrations of* *structures, where in some circumstances many modes are not excited, this being true for all times due to the existence of a small dissipation in the structure. A precise statement of this last assertion would be an interesting application of these results.*

Exercise 5.50. *Consider a system of the form (5.25) with $\mu \in \mathbb{R}$ and such that $\mathbf{R}(0,\mu) = 0$ for all μ. Further assume that the eigenvalues of $\mathbf{L}$ are all purely imaginary $\{\pm i\omega_j; j = 0, 1, \dots, r\}$, with $\pm i\omega_0$ simple eigenvalues and such that the following nonresonance condition is satisfied:*

$$n\omega_0 \neq \omega_j, \quad j = 1, \dots, r, \quad n \in \mathbb{Z}.$$

Set

$$u = A\zeta_0 + \overline{A\zeta_0} + \mathbf{\Phi}(A, \overline{A}, \mu) + v, \quad A \in \mathbb{C}, \ v \in E_1,$$

where ζ_0 is an eigenvector of $\mathbf{L}$ associated to the eigenvalue $i\omega_0$, E_1 is the spectral subspace associated to the eigenvalues $\{\pm i\omega_j; j = 1, \dots, r\}$, and $\mathbf{\Phi}$ is a polynomial in its arguments taking values in $\mathbb{R}^n$.

- *(i) Check that the hypotheses of Theorem 5.45 are satisfied, with $\mathbf{L}_0$ being the restriction of $\mathbf{L}$ to the spectral space associated to the eigenvalues $\pm i\omega_0$ and $\mathbf{L}_1$ the restriction of $\mathbf{L}$ to E_1.*
- *(ii) Show that there is a polynomial $\mathbf{\Phi}$ such that the system satisfied by $(A, \overline{A}, v)$ becomes*

$$\begin{aligned}\frac{dA}{dt} &= Ag(|A|^2, \mu) + R_0(A, \overline{A}, v, \mu) + \rho_0(A, \overline{A}, \mu) \\ \frac{dv}{dt} &= \mathbf{L}_1 v + \mathbf{R}_1(A, \overline{A}, v, \mu) + \boldsymbol{\rho}_1(A, \overline{A}, \mu),\end{aligned}$$

with the properties:

$$\begin{aligned}&g(|A|^2, \mu) = i\omega_0 + a\mu + b|A|^2 + h.o.t., \\ &|R_0(A, \overline{A}, v, \mu)| + \|\mathbf{R}_1(A, \overline{A}, v, \mu)\| = O(\|v\|(|A| + \|v\| + |\mu|)), \\ &\sup_{\|u_0\| + \|\mu\| \leq \delta} (|\rho_0(u_0, \mu)| + \|\boldsymbol{\rho}_1(u_0, \mu)\|) \leq Me^{-C/\delta^b}.\end{aligned}$$

- *(iii) Determine the first order terms of the polynomial $\mathbf{\Phi}$. (One finds the same formulas as for the Hopf bifurcation in Section 5.4.2.)*
- *(iv) Notice that if $b_r < 0$, and if at time $t = 0$ the v component is 0, or exponentially small, then it stays exponentially small for a very long time.*

6 Hydrodynamic Instabilities

6.1 Hydrodynamic Problem

Consider a viscous incompressible fluid filling a domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$. We present in this section the hydrodynamic problem corresponding to the following three types of domains:

(i) a smooth bounded domain $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$;

(ii) an infinite cylindrical domain $\Omega = \Sigma \times \mathbb{R}$, where the section Σ is a smooth bounded domain in $\mathbb{R}^2$;

(iii) a domain situated between two planes $\Omega = \mathbb{R}^2 \times I$, where $I = (\alpha, \beta)$ is a bounded interval in $\mathbb{R}$.

The velocity V of fluid particles and the pressure p are functions of $(x,t) \in \Omega \times \mathbb{R}^+$ and satisfy the Navier–Stokes equations

$$\begin{aligned} \frac{\partial V}{\partial t} + (V \cdot \nabla)V + \frac{1}{\rho}\nabla p &= \nu \Delta V + f(x), \\ \nabla \cdot V &= 0. \end{aligned} \tag{6.1}$$

In this system $V(x,t)$ has two or three components, when $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$, respectively, the volumic mass ρ is constant, ∇, $\nabla\cdot$, and Δ denote the gradient, divergence and Laplace operators, respectively, ν is the kinematic viscosity, and f represents an external massic force, independent of t. The first equation represents the momentum balance, while the second is the incompressibility condition.

Boundary Conditions System (6.1) is completed by boundary conditions. In the three cases, we assume that we have fixed geometric boundaries.

The simplest situation occurs in case (i) of a smooth bounded domain Ω, when the boundary conditions are

$$V|_{\partial\Omega} = a, \quad \int_{\partial\Omega} a \cdot n \, dS = 0, \tag{6.2}$$

where a is a given vector field, independent of t and having zero total flux, in order to be compatible with the incompressibility condition, and n is the exterior unit normal to $\partial\Omega$.

In case (ii) of a cylindrical domain $\Omega = \Sigma \times \mathbb{R}$, the boundary conditions are

$$V|_{\partial\Sigma\times\mathbb{R}} = a, \quad \int_{\partial\Sigma} a \cdot n \, ds = 0, \tag{6.3}$$

to which one can add, for instance, the following periodicity conditions along the cylinder:

$$V(x,t) = V(x + he_z, t), \quad \nabla p(x,t) = \nabla p(x + he_z, t) \text{ for all } x = (X, z) \in \Sigma \times \mathbb{R}, \tag{6.4}$$

where h is the period in the direction $z \in \mathbb{R}$ along the cylinder, and $e_z = (0,1) \in \Sigma \times \mathbb{R}$. Notice that we require only ∇p to be periodic and not p, which would also be a possibility, but less realistic. These conditions are completed by the assumption

$$\int_{\Sigma} V \cdot n \, dS = D, \tag{6.5}$$

where D is a given constant, showing that V has a given flux through the section Σ of the cylinder. It is not difficult to check that this flux is independent of $z \in \mathbb{R}$. This implies that p is allowed to increase linearly in z over a period.

Finally, in case (iii) of a domain $\Omega = \mathbb{R}^2 \times (\alpha, \beta)$ situated between two planes, the boundary conditions are

$$V|_{\mathbb{R}^2\times\{\alpha\}} = V|_{\mathbb{R}^2\times\{\beta\}} = a,$$

which imply that the total mass flux through the periodicity domain is zero, together with a biperiodicity condition,

$$V(x,t) = V(x + n_1e_1 + n_2e_2, t), \quad \nabla p(x,t) = \nabla p(x + n_1e_1 + n_2e_2, t) \tag{6.6}$$

for all $x = (X, z) \in \mathbb{R}^2 \times (\alpha, \beta)$, where $(n_1, n_2) \in \mathbb{Z}^2$, and the lattice of periods is generated by two noncolinear vectors e_1 and e_2 in $\mathbb{R}^2$. To these conditions we add two conditions on the flux of the velocity in the directions of two vectors k_1 and k_2 in the X-plane,

$$\int_{\Sigma_1} V \cdot k_2\, dS = D_1, \quad \int_{\Sigma_2} V \cdot k_1\, dS = D_2. \tag{6.7}$$

The vectors k_1 and k_2 are such that

$$\langle e_j, k_l \rangle = 2\pi\delta_{jl}, \tag{6.8}$$

and Σ_1 (resp., Σ_2) is the face orthogonal to k_2 (resp., to k_1) of the parallelepiped built with vectors e_1, e_2 and the interval (α, β) orthogonally to the X-plane, which constitutes the domain of periodicity.

Remark 6.1 (Free boundaries)**.** *Sometimes the boundary, or part of the boundary, of the domain Ω is "free," which means that the fluid is in contact with another fluid, the common boundary being unknown. Here, we only mention the simplified situation in which one assumes that the part of the boundary $\partial\Omega_1$, say, where the fluid is in contact with another fluid, is fixed. (This is acceptable for instance if the external fluid is mercury and the internal one is water.) Then, on this part of the boundary one has the following conditions:*

$$V \cdot n|_{\partial\Omega_1} = 0, \tag{6.9}$$

showing that no fluid crosses the boundary, and

$$(\nabla V + \nabla^t V) \cdot n|_{\partial\Omega_1} \times n = 0, \tag{6.10}$$

showing that the tangent stresses cancel.

Basic Solution We assume that a smooth stationary solution $(V^{(0)}(x), p^{(0)}(x))$ is known for system (6.1), together with the corresponding boundary conditions. We set

$$V = V^{(0)} + U, \quad p = p^{(0)} + \rho q,$$

which leads to the system

$$\begin{aligned} \frac{\partial U}{\partial t} &= \nu\Delta U - (V^{(0)} \cdot \nabla)U - (U \cdot \nabla)V^{(0)} - (U \cdot \nabla)U - \nabla q \\ \nabla \cdot U &= 0. \end{aligned} \tag{6.11}$$

In case (i) the boundary condition (6.2) becomes

$$U|_{\partial\Omega} = 0, \tag{6.12}$$

whereas in case (ii), the boundary conditions (6.3), (6.4), and (6.5) become, respectively,

$$U|_{\partial\Sigma\times\mathbb{R}} = 0, \tag{6.13}$$

$$U(x,t) = U(x + he_z, t), \quad \nabla q(x,t) = \nabla q(x + he_z, t) \text{ for all } x = (X, z) \in \Sigma \times \mathbb{R},$$

and

$$\int_\Sigma U \cdot n \, dS = 0. \tag{6.14}$$

The boundary conditions in case (iii) are similar.

Analytical Set-up We introduce now the basic Hilbert spaces in which system (6.11), together with the corresponding boundary conditions, is analyzed.

In case (i), we restrict to the case $\Omega \subset \mathbb{R}^3$, and define the Hilbert space

$$\mathcal{X} = \left\{ U \in \left(L^2(\Omega)\right)^3 \; ; \; \nabla \cdot U = 0, \; U \cdot n|_{\partial\Omega} = 0 \right\},$$

equipped with the scalar product of $\left(L^2(\Omega)\right)^3$. Notice that here the trace $U \cdot n|_{\partial\Omega}$ is well-defined in $H^{-1/2}(\partial\Omega)$ (e.g., see [68]). Next, we consider the subspace

$$\mathcal{Z} = \left\{ U \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^3 \; ; \; \nabla \cdot U = 0 \right\} \subset \mathcal{X};$$

i.e., the functions in this subspace satisfy the boundary condition (6.12).

A key property of the Hilbert space $\mathcal{X}$ is that the kernel of the orthogonal projection $\mathbf{\Pi}_0$ in $\left(L^2(\Omega)\right)^3$ on the subspace $\mathcal{X}$ can be identified with the space $\{\nabla\phi \; ; \; \phi \in H^1(\Omega)\}$ (e.g., see [77, 45, 68]). Then, using the projection $\mathbf{\Pi}_0$, the pressure term ∇q in (6.11) can be eliminated, and we obtain a system of the form

$$\frac{dU}{dt} = \mathbf{L}U + \mathbf{R}(U) \tag{6.15}$$

posed in $\mathcal{X}$ for $U(\cdot, t) \in \mathcal{Z}$, where

$$\mathbf{L}U = \mathbf{\Pi}_0 \left(\nu\Delta U - (V^{(0)} \cdot \nabla)U - (U \cdot \nabla)V^{(0)} \right), \quad \mathbf{R}(U) = -\mathbf{\Pi}_0 \left((U \cdot \nabla)U\right). \tag{6.16}$$

The linear operator $\mathbf{L}$, acting in $\mathcal{X}$, may be regarded as a lower order perturbation of the self-adjoint operator $\mathbf{\Pi}_0(\nu\Delta U)$. It is a closed operator in $\mathcal{X}$, with dense domain $\mathcal{Z}$ and a compact resolvent. The spectrum of $\mathbf{L}$ consists of isolated eigenvalues with finite multiplicities, situated in a sector of the complex plane centered on the real axis, and oriented on the negative side of this axis [77]. Its resolvent satisfies the estimate (4.3) (see [77, 45]). Actually, one can prove in this case that $\mathbf{L}$ is the generator of an analytic semigroup $e^{\mathbf{L}t}$ for $t > 0$ (see [39]).

The nonlinear term $\mathbf{R}(U)$ satisfies $\mathbf{R}(U) \in \mathcal{X} \cap \left(H^1(\Omega)\right)^3$ for $U \in \mathcal{Z}$, by the Sobolev embedding theorem, and the map $\mathbf{R} : \mathcal{Z} \to \mathcal{X}$ is quadratic and continuous.

Remark 6.2. *In the case of a free boundary, when a part of the boundary is subjected to conditions (6.9)–(6.10), we can use the same space $\mathcal{X}$, and replace $\left(H_0^1(\Omega)\right)^3$ in the definition of $\mathcal{Z}$ by the space*

$$\left\{U \in \left(H^1(\Omega)\right)^3 \; ; \; U|_{\partial\Omega_2} = 0, \; U \cdot n|_{\partial\Omega_1} = 0, \; (\nabla U + \nabla^t U) \cdot n\}|_{\partial\Omega_1} \times n = 0\right\},$$

where $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$.

In case (ii) of a cylindrical domain $\Omega = \Sigma \times \mathbb{R}$, we define the Hilbert space

$$\mathcal{X} = \left\{U \in \left(L^2(\Sigma \times (\mathbb{R}/h\mathbb{Z}))\right)^3 \; ; \; \nabla \cdot U = 0, \; U \cdot n|_{\partial\Sigma\times\mathbb{R}} = 0, \; \int_\Sigma U \cdot n \, dS = 0\right\}.$$

We point out that here the orthogonal complement of $\mathcal{X}$ in $\left(L^2(\Sigma \times (\mathbb{R}/h\mathbb{Z}))\right)^3$ is the space $\{\nabla\phi \; ; \; \phi \in H^1(\Sigma \times (\mathbb{R}/h\mathbb{Z})) + z\mathbb{R}\}$, i.e., $\nabla\phi$ is a periodic function, while ϕ is not periodic [8]. The space $\mathcal{Z}$ is defined as a subspace of $\left(H^2(\Sigma \times (\mathbb{R}/h\mathbb{Z}))\right)^3 \cap \mathcal{X}$, according to the boundary conditions. Using again the orthogonal projection $\mathbf{\Pi}_0$ on $\mathcal{X}$, the Navier–Stokes system can be written in form (6.15) with $\mathbf{L}$ and $\mathbf{R}$ defined as in (6.16).

Similarly, in case (iii) for a domain $\Omega = \mathbb{R}^2 \times I$, we can define the spaces $\mathcal{X}$ and $\mathcal{Z}$ in an appropriate manner taking into account the boundary conditions, and use the orthogonal projection $\mathbf{\Pi}_0$ to write the system in the form (6.15). In both cases, the properties of $\mathbf{L}$ and $\mathbf{R}$ mentioned above are still valid.

Summarizing, in the three cases we have a system of the form (6.15), for which Hypotheses 4.1 and 4.6 required by the center manifold theorem in section 4 are verified, and in order to check Hypothesis 4.4, it is enough to locate the eigenvalues that have the largest real parts. In general, this is obtained by a careful study of their location for each specific physical situation. We point out that the parameter dependency comes from the viscosity ν and the boundary data, which influence the basic solution $(V^{(0)}, p^{(0)})$.

We present in the next two subsections two classical examples where the theoretical tools developed in the previous chapters apply particularly well. A description of classical experiments and physical results connected to both examples may be found in the books [38, 44].

6.2 Couette–Taylor Problem

We briefly present in this section some results on the Couette–Taylor problem, which have been obtained with the help of the methods described in this book. We refer to the book [8] for details, and to [67] for the huge bibliography on this problem.

Hydrodynamic Problem Consider two coaxial cylinders of radii R_1 (the inner cylinder), and R_2 (the outer cylinder), the gap between them being filled by an incompressible viscous fluid. Both cylinders rotate with constant rotation rates Ω_1 and Ω_2, respectively (see Figure 6.1(i)). For fixing ideas, we assume that $\Omega_1 > 0$. When the length of the cylinders is large with respect to the gap $R_2 - R_1$, it is physically reasonable, for a first study, to replace the rather complicated physically relevant boundary conditions at the ends of the cylinders by periodicity conditions, as this is also suggested by experimental observations.

The mathematical problem consists then in solving the Navier–Stokes system (6.1) in the cylindrical domain

$$\Omega = \Sigma \times \mathbb{R}, \quad \Sigma = \{(x,y) \in \mathbb{R}^2 \ ; \ R_1^2 < x^2 + y^2 < R_2^2\},$$

with $f = 0$ and the boundary conditions (6.3), (6.4), (6.14). In these boundary conditions a is now the velocity $R_1\Omega_1$ or $R_2\Omega_2$ tangent to the inner or outer cylinder, respectively, and orthogonal to the axis of rotation, and the flux of the velocity through any section is $D = 0$.

Couette Flow This problem possesses a basic steady solution $(V^{(0)}, p^{(0)})$, *the Couette flow*, given in cylindrical coordinates (r, θ, z) by

$$V^{(0)} = (0, v_0(r), 0), \quad p^{(0)} = \rho \int \frac{v_0^2}{r}\, dr$$

with

$$v_0(r) = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}\, r + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}\, \frac{1}{r}.$$

Notice that this solution is independent of z, the coordinate along the cylinder, and θ, the angle around the axis, and that its streamlines are circles centered on the rotation axis.

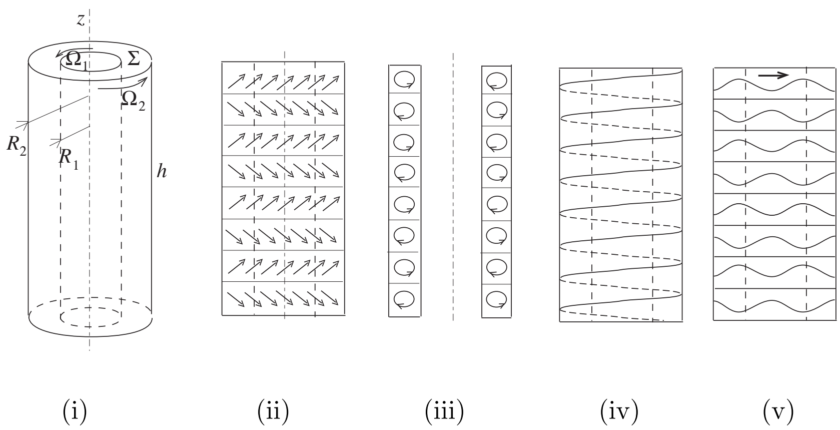


Figure 6.1: (i) Domain of periodicity for the Couette–Taylor problem. (ii) Side view of the Taylor vortex flow. (iii) Meridian view of the Taylor cells. (iv) Helicoidal waves (traveling in both z and θ directions). (v) Ribbons (standing in z direction, traveling in θ direction).

Symmetries A fundamental feature of this system consists in its symmetries. When $f = 0$, the Navier–Stokes system (6.1) possesses the Galilean invariance, which is typical to any physical system ruled by Newtonian laws. The result is the symmetries of the system are restricted to the symmetries of the boundary conditions. For the Couette–Taylor problem, the invariance under translations along the z-axis allied with the periodicity conditions, and the invariance under reflections through any plane orthogonal to this axis induce an $O(2)$ symmetry (the same as in the example in Section 4.4.2). Notice that gravity plays no role here, since it may be included in the gradient of the pressure. In addition, the system is invariant under rotations around the z-axis that induce a $SO(2)$-symmetry.

In cylindrical coordinates (r, θ, z), we have the following linear representations of these symmetries:

$$\begin{aligned}
(\boldsymbol{\tau}_a V)(r, \theta, z) &= V(r, \theta, z + a), \quad a \in \mathbb{R}/h\mathbb{Z}, \\
(\mathbf{S}V)(r, \theta, z) &= (V_r(r, \theta, -z), V_\theta(r, \theta, -z), -V_z((r, \theta, -z)), \\
(\mathbf{R}_\phi V)(r, \theta, z) &= V(r, \theta + \phi, z), \quad \phi \in \mathbb{R}/2\pi\mathbb{Z},
\end{aligned}$$

which satisfy

$$\boldsymbol{\tau}_a \mathbf{S} = \mathbf{S}\boldsymbol{\tau}_{-a}, \quad \boldsymbol{\tau}_h = \mathbb{I}, \quad \boldsymbol{\tau}_a \boldsymbol{\tau}_b = \boldsymbol{\tau}_{a+b}.$$

Consequently, $(\boldsymbol{\tau}_a, \mathbf{S})$ is an $O(2)$ grouprepresentation, and $\mathbf{R}_\phi$ represents a $SO(2)$ action, which commutes with the $O(2)$ action. We point out that the basic Couette flow $(V^{(0)}, p^{(0)})$ is left invariant by all these symmetries, which are then inherited by the system (6.11).

Instabilities As usual in any physical problem, we need to choose the scales. Here the length scale is $(R_2 - R_1)$ and the velocity scale is $R_1\Omega_1$. Three dimensionless parameters appear in the equations of the problem, which we can choose as

$$\Omega_r = \frac{\Omega_2}{\Omega_1}, \quad \eta = \frac{R_1}{R_2}, \quad \mathcal{R} = \frac{R_1\Omega_1(R_2 - R_1)}{\nu},$$

where $\mathcal{R}$ is a Reynolds number. Consider the system (6.11) satisfied by perturbations of the basic Couette flow, and more precisely its formulation (6.15) as a first order system. Fixing the parameters Ω_r and η, we take $\mathcal{R}$ as bifurcation parameter, and denote the linear operator $\mathbf{L}$ in (6.15) by $\mathbf{L}_\mathcal{R}$. It turns out that the spectrum of $\mathbf{L}_\mathcal{R}$ is strictly contained in the left half-complex plane, i.e., the Couette flow is stable, for low values of $\mathcal{R}$, i.e. for small rotation rate of the inner cylinder, or high viscosity. Instabilities are obtained by increasing $\mathcal{R}$ (for instance by increasing the rotation rate of the inner cylinder). This may be interpreted by the fact that for Ω_1 large enough, the excess of centrifugal forces acting on particles close to the inner cylinder, with respect to those near the outer cylinder, becomes dominant if we diminish the viscosity ν. The nature of these instabilities now depends upon the values of Ω_r.

The Case $\Omega_r > 0$ *or* $\Omega_r < 0$ *Close to* 0 In this case it has been shown numerically that as $\mathcal{R}$ increases, there is a critical value $\mathcal{R}_c$ for which an eigenvalue of $\mathbf{L}_\mathcal{R}$ crosses the imaginary axis, passing through 0 from the left to the right, and all other eigenvalues remain in the left half-complex plane. We are here in the presence of a steady $O(2)$ bifurcation in which 0 is a double eigenvalue with complex conjugated eigenvectors

$$\zeta = e^{ik_c z}\widehat{U}(r), \quad \overline{\zeta} = \mathbf{S}\zeta,$$

where the wave number k_c is such that there is an integer n with

$$k_c h = 2n\pi,$$

and

$$\boldsymbol{\tau}_a\zeta = e^{ik_ca}\zeta \text{ for all } a \in \mathbb{R}.$$

Applying the center manifold Theorems 4.18 and 4.29, one finds a two-dimensional center manifold, and the reduced vector field commutes with the restrictions of $\boldsymbol{\tau}_a$ and $\mathbf{S}$ on the two-dimensional subspace $\mathcal{E}_0$ spanned by ζ and $\overline{\zeta}$. We point out that $\mathbf{R}_\phi$ acts trivially on $\mathcal{E}_0$, which means that all solutions on the center manifold are invariant under $\mathbf{R}_\phi$. Consequently, for the reduced system we are in the situation described in Section 3.4 (see also the first part of example in Section 4.4.2). The reduced dynamics are ruled by the following ordinary differential equation:

$$\frac{dA}{dt} = Ag(|A|^2,\mu), \quad g(|A|^2,\mu) = a\mu + b|A|^2 + h.o.t., \quad a,b \in \mathbb{R}, \tag{6.17}$$

in which A is complex-valued, $\mu = \mathcal{R} - \mathcal{R}_c$, and a, b are real numbers depending upon Ω_r. Equation (6.17) is called the *Landau equation* in the physics literature, as it was first formally derived by Landau [46].

According to the results in [8] the coefficients a and b are such that $a > 0$, $b < 0$ when $\Omega_r > 0$, and b changes sign for a certain small value of $\Omega_r < 0$. We can now apply Theorem 3.17 in section 2 and conclude that for $b < 0$ (resp., for $b > 0$) we have a *supercritical (resp., subcritical) pitchfork bifurcation to a circle of steady stable (resp., unstable) solutions.* In the infinite-dimensional phase space of the full system (6.15), this circle of solutions corresponds to solutions that are shifted along the z direction, i.e., obtained by the action of $\boldsymbol{\tau}_a$. In addition, the action of $\boldsymbol{\tau}_{2\pi/k_c}$ is trivial, which means that the period in z of the bifurcating solutions is $2\pi/k_c = h/n$, and the solutions are invariant under the action of $\mathbf{R}_\phi$. Two of the shifted solutions are also invariant under $\mathbf{S}$, which means that the corresponding flow does not cross the planes $z = k\pi/k_c$, $k \in \mathbb{Z}$, thus forming axisymmetric toroidal cells. This constitutes the *Taylor vortex flow* (see Figure 6.1(ii)–(iii)).

The Case $\Omega_r < 0$, not too close to 0 In this case, numerical results show that the Couette flow first becomes unstable at a critical value $\mathcal{R}_c$ of $\mathcal{R}$, when a pair of complex conjugate eigenvalues of $\mathbf{L}_\mathcal{R}$ crosses the imaginary axis, from the left to the right, as $\mathcal{R}$ is increased, and the rest of the spectrum stays in the left half-complex plane. These two eigenvalues are both double, as this case is generic for $O(2)$ equivariant systems, with two eigenvectors of the form

$$\zeta_0 = e^{i(k_cz+m\theta)}\widehat{U}(r), \quad \zeta_1 = e^{i(-k_cz+m\theta)}\mathbf{S}\widehat{U}(r),$$

where $m \neq 0$, and the critical wave number k_c is determined as in the previous case.

Applying the center manifold Theorems 4.18 and 4.29, we find a four-dimensional center manifold, and the reduced vector field commutes with the actions of the induced symmetries $\boldsymbol{\tau}_a$, $\mathbf{S}$, and $\mathbf{R}_\phi$, found from

$$\boldsymbol{\tau}_a\zeta_0 = e^{ik_ca}\zeta_0, \quad \boldsymbol{\tau}_a\zeta_1 = e^{-ik_ca}\zeta_1, \quad \mathbf{S}\zeta_0 = \zeta_1, \quad \mathbf{S}\zeta_1 = \zeta_0,$$
$$\mathbf{R}_\phi\zeta_0 = e^{im\phi}\zeta_0, \quad \mathbf{R}_\phi\zeta_1 = e^{im\phi}\zeta_1.$$

We are here in the presence of a Hopf bifurcation with $O(2)$ symmetry, as discussed in Section 5.4.4, but with an additional $SO(2)$ symmetry represented by $\mathbf{R}_\phi$. With the notations from Section 5.4.4, it turns out that the dynamics are ruled by a system in $\mathbb{C}^2$ of the form

$$\begin{aligned}\frac{dA}{dt} &= AP(|A|^2, |B|^2, \mu) \\ \frac{dB}{dt} &= BP(|B|^2, |A|^2, \mu),\end{aligned}$$

where $\mu = \mathcal{R} - \mathcal{R}_c$, and

$$P(|A|^2, |B|^2, \mu) = i\omega + a\mu + b|A|^2 + c|B|^2 + h.o.t.$$

is a smooth function of its arguments, and with no "remainder ρ."

The coefficients a, b, and c are complex, and their explicit values can be found in [8]. The bifurcating solutions corresponding to $A = 0$ or to $B = 0$ travel along and around the z-axis with constant velocities. These are *helicoidal waves*, also called *spirals*, and they are axially periodic just as the Taylor vortex flow (see Figure 6.1(iv)). The bifurcating solutions obtained for $|A| = |B|$ are *standing waves* located in fixed horizontal periodic cells, as they are for the Taylor vortex flow, but with a non-axisymmetric internal structure rotating around the axis with a constant velocity. These solutions are also called *ribbons* (see Figure 6.1(v)). We point out that both types of waves may be observed, depending upon the other parameters (see [8] for the predicted parameter values, and [67] for the corresponding experimental observations).

Further Bifurcations The next step consists in considering the circle of solutions corresponding to the Taylor vortex flow and to study the resulting bifurcation, which is a symmetry-breaking bifurcation. Here, one may proceed as indicated in Section 4.3.4, for systems possessing a continuous symmetry and a one-parameter family of equilibria. Theorem 4.34 applies, provided we know the "critical" eigenvalues, in addition to the eigenvalue 0, of the operator obtained by linearizing at one point of the "circle" of Taylor vortex solutions where the solution is invariant under symmetry $\mathbf{S}$. It is shown in [8] that when $\mathcal{R}$ passes a new critical value $\mathcal{R}_2$, depending on the parameters Ω_r and η, a Hopf bifurcation occurs. To one purely imaginary eigenvalue corresponds a non-axisymmetric eigenvector, which is either symmetric or antisymmetric, with the same or the double axial periodicity as the Taylor flow, and leading to *twisted vortices*, *wavy vortices*, *wavy inflow boundaries*, or *wavy outflow boundaries*. All these flows are rotating waves around the z-axis, due to the Hopf bifurcation with the $SO(2)$ symmetry broken by the eigenvectors (see also Section 5.3.1), but with various cell structures, the two first having the same axial periodicity as the Taylor vortex flow, the last two having a double period. One can proceed in the same way when starting with spirals or ribbons instead of the Taylor vortex flow [8].

Finally, we point out that these tools can also be used to study imperfect situations such as when cylinders are slightly eccentric, which breaks the $SO(2)$ symmetry, or in the presence of a little flux of fluid downwards, e.g., due to a leak in the apparatus, which breaks the reflection symmetry $\mathbf{S}$, or in the presence of a small bump on one cylinder, which breaks the translation invariance (see also the example in Section 4.4.2).

6.3 Bénard–Rayleigh Convection Problem

Hydrodynamic Problem

Consider a viscous fluid filling the region between two horizontal planes. Each planar boundary may be a rigid plane, or a "free" boundary in the sense explained in Remark 6.1. In addition, we assume that the lower and upper planes are at temperatures T_0 and T_1, respectively, with $T_0 > T_1$ (see Figure 6.2(i)). The difference of temperature between the two planes modifies the fluid density, tending to place the lighter fluid below the heavier one. The gravity then induces, through the Archimedian force, an instability of the "conduction regime" where the fluid is at rest, while the temperature depends linearly on the vertical coordinate z. This instability is prevented up to a certain level by viscosity, so that there is a critical value of the temperature difference, below which nothing happens and above which a "convective regime" appears.

The Navier–Stokes system (6.1) is not sufficient to describe this situation. An additional equation for energy conservation is needed, where the internal energy is proportional to temperature. In the Boussinesq approximation, the dependency of the density ρ in function of the temperature T,

$$\rho = \rho_0 \left(1 - \alpha(T - T_0)\right),$$

where α is the volume expansion coefficient, is taken into account in the momentum equation, only in the external volumic gravity force $-\rho g e_z$, introducing the coupling between (V, p) and T. We refer to [38, Vol. II] for a very complete discussion and bibliography on various geometries and boundary conditions in this problem.

Several different scalings are used in literature. We adopt here the one in [44], which consists in choosing the length, time, velocity, and temperature scales respectively as d, d^2/κ, κ/d, $\nu\kappa/\alpha g d^3$, where d is the distance between the planes, κ is the thermal diffusivity, and ν, α, and g are as above. This leads to the system

$$\begin{aligned} \frac{\partial V}{\partial t} + V \cdot \nabla V + \nabla p &= \mathcal{P}(\theta e_z + \Delta V) \\ \nabla \cdot V &= 0 \\ \frac{\partial \theta}{\partial t} + V \cdot \nabla \theta &= \Delta\theta + \mathcal{R}(V \cdot e_z), \end{aligned} \tag{6.18}$$

replacing the Navier–Stokes system (6.11). Here θ is the deviation of the temperature from the conduction profile, which satisfies the boundary conditions, and $V = (V_1, V_2, V_z)$, p, and θ are functions of (x, t), $x = (X, z)$, with $X = (x_1, x_2) \in \mathbb{R}^2$ the horizontal coordinates and $z \in (0, 1)$ the vertical coordinate, e_z being the unitary ascendent vector. There are two dimensionless numbers in this problem: the Prandtl number $\mathcal{P}$ and the Rayleigh number $\mathcal{R}$ defined respectively as

$$\mathcal{P} = \frac{\nu}{\kappa}, \quad \mathcal{R} = \frac{\alpha g d^3 (T_0 - T_1)}{\nu\kappa}.$$

System (6.18) is completed by the boundary conditions

$$V_z = \theta = 0, \quad z = 0, 1,$$

together with either a "rigid surface" condition

$$V_1 = V_2 = 0, \tag{6.19}$$

or a "free surface" condition

$$\frac{\partial V_1}{\partial z} = \frac{\partial V_2}{\partial z} = 0 \tag{6.20}$$

on the planes $z = 0$ or $z = 1$. Notice that here the kinematic viscosity is independent of the temperature T. If this is not the case, some qualitative results change. Also, adding a solute with a certain concentration, satisfying an equation and boundary conditions of the same form as θ, gives richer results [38, Vol. II].

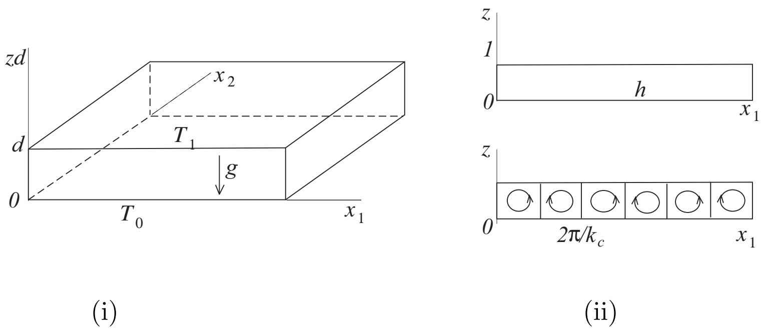


Figure 6.2: (i) Bénard–Rayleigh problem. (ii) Domain of periodicity for bidimensional convection (above) and convection rolls (below).

Bidimensional Convection

We restrict ourselves first to the case of bidimensional flows, i.e., we assume that $V_2 = 0$, and $V = (V_1, V_z)$, p, and θ are only functions of x_1, z, and t.

Formulation as a First Order System We set $U = (V, \theta)$, and then the system is of the form (6.15) in the space $\mathcal{X}$ of h-periodic functions in x_1, defined by

$$\mathcal{X} = \left\{ U \in \left(L^2((\mathbb{R}/h\mathbb{Z}) \times (0,1))\right)^3 \; ; \; \nabla \cdot V = 0, \; V_z|_{z=0,1} = 0, \; \int_0^1 V_1 dz = 0 \right\}.$$

In the case of rigid boundary conditions (6.19) on both planes $z = 0$ and $z = 1$, the domain of $\mathbf{L}$ is defined by

$$\mathcal{Z}_{(r,r)} = \left\{ U \in \left(H^2((\mathbb{R}/h\mathbb{Z}) \times (0,1))\right)^3 \; ; \; \nabla \cdot V = 0, \right.$$
$$\left. V|_{z=0,1} = \theta|_{z=0,1} = 0, \; \int_0^1 V_1 dz = 0 \right\},$$

and similarly we define $\mathcal{Z}_{(r,f)}$, $\mathcal{Z}_{(f,r)}$, and $\mathcal{Z}_{(f,f)}$ by replacing the rigid boundary condition $V_1 = 0$ by the free boundary condition $\partial V_1/\partial z = 0$ on $z = 1$, $z = 0$, and $z = 0, 1$, respectively (see Figure 6.2(ii)). Here we have

$$\mathbf{L}U = (\mathbf{\Pi}_0 \mathcal{P}(\Delta V + \theta e_z), \Delta\theta + \mathcal{R} V_z), \quad \mathbf{R}(U) = (-\mathbf{\Pi}_0(V \cdot \nabla V), -V \cdot \nabla\theta), \tag{6.21}$$

with $\mathbf{R} : \mathcal{Z} \to \mathcal{Y} = \mathcal{X} \cap \big(H^1((\mathbb{R}/h\mathbb{Z}) \times (0,1))\big)^3$ quadratic and continuous. Here $\mathcal{Z}$ represents one of the spaces $\mathcal{Z}_{(r,r)}$, $\mathcal{Z}_{(r,f)}$, $\mathcal{Z}_{(f,r)}$, and $\mathcal{Z}_{(f,f)}$ above, depending upon the choice of boundary conditions. Notice that the pressure p is not necessarily periodic in x_1, and that the orthogonal projection $\mathbf{\Pi}_0$ in $\big(L^2((\mathbb{R}/h\mathbb{Z}) \times (0,1))\big)^3$ on the subspace $\mathcal{X}$ eliminates the periodic gradient ∇p, as in Section 6.1.

A specific property of $\mathbf{L}$ in this case is that there is a special scalar product in the Hilbert space $\mathcal{X}$, with corresponding norm equivalent to the usual one, such that $\mathbf{L}$ is *self-adjoint.* This scalar product is defined by

$$\langle U^{(1)}, U^{(2)} \rangle = \langle V^{(1)}, V^{(2)} \rangle|_{(L^2((\mathbb{R}/h\mathbb{Z})\times(0,1)))^2} + \frac{\mathcal{P}}{\mathcal{R}} \langle \theta^{(1)}, \theta^{(2)} \rangle|_{L^2((\mathbb{R}/h\mathbb{Z})\times(0,1))}.$$

As a consequence, the spectrum of $\mathbf{L}$ is now located on the real axis. Notice that $\mathbf{L}$ is a relatively compact perturbation of the uncoupled self-adjoint negative operator

$$\mathbf{L}'U = (\mathbf{\Pi}_0 \mathcal{P} \Delta V, \Delta\theta),$$

and that it has a compact resolvent, since its domain is compactly embedded in $\mathcal{X}$ (see [39]). The spectrum of $\mathbf{L}$ consists then of isolated semisimple real eigenvalues of finite multiplicities, accumulating at $-\infty$, only. Furthermore, the resolvent estimate (4.3) is straightforward. As for the case considered in Section 6.1, the hypotheses required by the center manifold theorem are all satisfied.

Symmetries This problem is invariant under translations parallel to the x_1-axis and under the reflection $x_1 \mapsto -x_1$. Then the system (6.15) possesses an $O(2)$ symmetry group represented by $\boldsymbol{\tau}_a$ and $\mathbf{S}$ defined through

$$\begin{aligned} (\boldsymbol{\tau}_a U)(x_1, z) &= U(x_1 + a, z), \quad a \in \mathbb{R}/h\mathbb{Z} \\ (\mathbf{S}U)(x_1, z) &= (-V_1(-x_1, z), V_z(-x_1, z), \theta(-x_1, z)), \end{aligned} \tag{6.22}$$

where $\boldsymbol{\tau}_h = \mathbb{I}$, because of the periodicity assumption. In addition, in the cases of "rigid-rigid" and "free-free" boundary conditions, i.e., with $\mathcal{Z}_{(r,r)}$ and $\mathcal{Z}_{(f,f)}$, respectively, there is the additional symmetry with respect to the half-plane $z = 1/2$,

$$(\mathbf{S}_z U)(x_1, z) = (V_1(x_1, 1-z), -V_z(x_1, 1-z), -\theta(x_1, 1-z)). \tag{6.23}$$

Bifurcations We fix the Prandtl number $\mathcal{P}$ and take the Reynolds number $\mathcal{R}$ as bifurcation parameter. As before, we denote by $\mathbf{L}_{\mathcal{R}}$ the linear operator $\mathbf{L}$ in (6.15). Then upon increasing $\mathcal{R}$ from 0, there is a critical value $\mathcal{R}_c$ for which the largest real eigenvalue of $\mathbf{L}_{\mathcal{R}}$ crosses the imaginary axis from the left to the right [65, 69] (see also [38, Vol. II]). The eigenvalue 0 of $\mathbf{L}_{\mathcal{R}_c}$ is double, as it is generic for $O(2)$ equivariant operators, and the corresponding eigenvectors are of the form

$$\zeta = e^{ik_c x_1} \widehat{U}(z), \quad \overline{\zeta} = \mathbf{S}\zeta,$$

where k_c is a positive critical wavenumber. In the case of "free-free" boundary conditions, the eigenvectors are explicit and k_c is easily obtained. In other cases, the existence of such a positive k_c may be proved analytically [72] (or following the method in [74]); see also [38, Vol. II], but its uniqueness is, so far, only a numerical evidence. Notice that the action of $\boldsymbol{\tau}_a$ on the eigenvector ζ is

$$\boldsymbol{\tau}_a \zeta = e^{ik_c a}\zeta,$$

so that we are in the presence of a steady bifurcation with $O(2)$ symmetry.

Applying the center manifold Theorems 4.18 and 4.29 , we find a two-dimensional center manifold and a reduced system, which commutes with the restrictions of $\boldsymbol{\tau}_a$ and $\mathbf{S}$ on the two-dimensional subspace $\mathcal{E}_0$ spanned by ζ and $\overline{\zeta}$. The reduced equation is a Landau equation (6.17), and we find precisely the situation described in Section 3.4 (see also the first part of the example in Section 4.4.2). Here $\mu = \mathcal{R} - \mathcal{R}_c$, and $a > 0$, $b < 0$ ([69, 75]; see also [38, Vol. II]). Notice that in the cases of "rigid-rigid" or "free-free" boundary conditions, the reduced system also commutes with the restriction on $\mathcal{E}_0$ of the symmetry $\mathbf{S}_z$. However, the action of this symmetry is $\pm\mathbb{I}$ on $\mathcal{E}_0$, which does not influence the Landau equation, already odd in $(A, \overline{A})$. Applying Theorem 3.17, we find a pitchfork bifurcation of a "circle" of stable steady solutions, obtained by translating with $\boldsymbol{\tau}_a$ a symmetric, periodic solution. All these solutions have the period $2\pi/k_c$ and, as in the previous section, appear in cells of size π/k_c, the velocity being tangent to the boundaries of the rectangular cells. These solutions are the *convection rolls* (see Figure 6.2(iii)).

Tridimensional Convection

Consider now the three-dimensional case, in which V_2 is not identically 0, and V, p, and θ are functions of X, z, and t, $X = (x_1, x_2)$. Here, we assume the biperiodicity condition (6.6), where the lattice of periods Γ is generated by two independent horizontal vectors $\{e_1, e_2\}$, and the dual lattice of wave vectors is generated by the two vectors $\{k_1, k_2\}$ defined by (6.8). It turns out that in this case the critical wavenumber found in the bidimensional case, is now the radius of a critical circle in the Fourier plane. It was shown in [41] that the only possible forms of periodic patterns are rolls, hexagons, regular triangles, and rectangles (see also [18]). Since experimental evidence mostly show convection in rolls and convection in hexagonal cells, we choose a lattice compatible with both patterns, as initiated in [63].

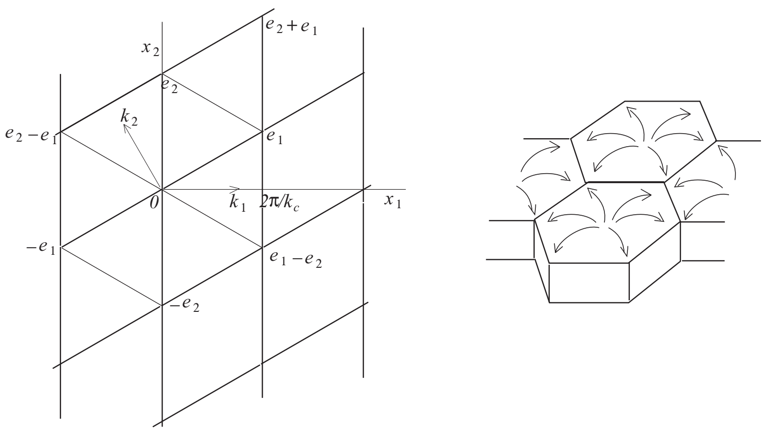
Formulation as a First Order System We choose

$$e_1 = h\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad e_2 = h(0,1), \quad k_1 = k_c(1,0), \quad k_2 = k_c\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

where h is determined by the critical wavelength k_c,

$$hk_c = \frac{4\pi}{\sqrt{3}}.$$

It is not difficult to check that this lattice is invariant under rotations of angle $\pi/3$ (see Figure 6.3(i)).



(i) (ii)

Figure 6.3: (i) Lattice Γ in the X-plane, for 3-D convection. (ii) Flow in a hexagonal cell.

According to the flux conditions (6.7), we choose the Hilbert spaces

$$\mathcal{X} = \Big\{ U \in \left(L^2((\mathbb{R}^2/\Gamma)\times(0,1))\right)^4 \ ; \ \nabla\cdot V = 0,\ V_z|_{z=0,1} = 0, \\ \int_{\Sigma_1} V\cdot k_2 dS = \int_{\Sigma_2} V\cdot k_1 dS = 0 \Big\},$$

and in the case of "rigid-rigid" boundary conditions

$$\mathcal{Z}_{(r,r)} = \Big\{ U \in \left(H^2((\mathbb{R}^2/\Gamma)\times(0,1))\right)^4 \ ; \ \nabla\cdot V = 0,\ V|_{z=0,1} = \theta|_{z=0,1} = 0, \\ \int_{\Sigma_1} V\cdot k_2 dS = \int_{\Sigma_2} V\cdot k_1 dS = 0 \Big\},$$

and similarly $\mathcal{Z}_{(r,f)}$, $\mathcal{Z}_{(f,r)}$, and $\mathcal{Z}_{(f,f)}$, by replacing the rigid boundary conditions $V_1 = V_2 = 0$ by the free boundary conditions $\partial V_1/\partial z = \partial V_2/\partial z = 0$ on $z = 1$, $z = 0$, and $z = 0, 1$, respectively. We set $U = (V, \theta)$, just as in the two-dimensional case, and then the system is of the form (6.15), with $\mathbf{L}$ and $\mathbf{R}$ defined as in (6.21). The linear operator $\mathbf{L}$ and the quadratic map $\mathbf{R}$ have the same properties as in the two-dimensional case.

Symmetries This problem is invariant under horizontal translations, represented by the operators $\boldsymbol{\tau}_a$ when replacing $x_1 + a$ by $X + a$ for any $a \in \mathbb{R}^2/\Gamma$, and invariant under the mirror symmetry $\mathbf{S}$ defined as in (6.22). In addition, it is invariant under the rotation

$$(\mathbf{R}_{2\pi/3}U)(X,z) = \left(R_{2\pi/3}(V(R_{-2\pi/3}X,z)), \theta(R_{-2\pi/3}X,z)\right), \tag{6.24}$$

where $R_{2\pi/3}$ is the horizontal rotation, in the X-plane, of angle $2\pi/3$. The group generated by $\mathbf{S}$ and $\mathbf{R}_{2\pi/3}$ is denoted by D_6, consisting of rotations on a circle of angle $\pi/3$ together with the symmetries through a diameter. In the cases of "rigid-rigid" and "free-free" boundary conditions, we still have the symmetry $\mathbf{S}_z$, defined by (6.23) with x_1 replaced by X.

Bifurcations We fix the Prandtl number $\mathcal{P}$ and take the Reynolds number $\mathcal{R}$ as bifurcation parameter. As before, we denote by $\mathbf{L}_{\mathcal{R}}$ the linear operator $\mathbf{L}$ in (6.15). Upon increasing $\mathcal{R}$, there is a critical value $\mathcal{R}_c$ for which the largest real eigenvalue of $\mathbf{L}_{\mathcal{R}}$ crosses the imaginary axis from the left to the right, which is now of multiplicity six. The associated eigenvectors are now of the form

$$\zeta_j = e^{ik_j \cdot X}\widehat{U}_j(z), \quad j = 1, \ldots, 6,$$

and satisfy

$$\zeta_2 = \mathbf{R}_{2\pi/3}\zeta_1, \quad \zeta_3 = \mathbf{R}_{-2\pi/3}\zeta_1, \quad \zeta_{j+3} = \mathbf{S}\zeta_j = \overline{\zeta_j}, \quad j = 1, 2, 3,$$

where

$$k_3 = -(k_1 + k_2), \quad k_{j+3} = -k_j, \quad j = 1, 2, 3.$$

Furthermore

$$\boldsymbol{\tau}_a\zeta_j = e^{ik_j \cdot a}\zeta_j, \quad e^{ik_3 \cdot a} = e^{-i(k_1+k_2)\cdot a},$$

and the action of the symmetry $\mathbf{S}_z$ is either the identity $\mathbb{I}$ or $-\mathbb{I}$, when it is relevant.

Applying the center manifold Theorems 4.18 and 4.29, we find a six-dimensional center manifold. For $U_0 \in \mathcal{E}_0$, the eigenspace associated to the eigenvalue 0 of $\mathbf{L}_{\mathcal{R}_c}$, we set

$$U_0 = A\zeta_1 + B\zeta_2 + C\zeta_3 + \overline{A\zeta_1} + \overline{B\zeta_2} + \overline{C\zeta_3}, \tag{6.25}$$

and then we have the induced symmetries

$$\begin{aligned} \boldsymbol{\tau}_a(A, B, C) &= (Ae^{ik_1 \cdot a}, Be^{ik_2 \cdot a}, Ce^{ik_3 \cdot a}) \text{ for all } a \in \mathbb{R}^2/\Gamma, \\ \mathbf{S}(A, B, C) &= (\overline{A}, \overline{B}, \overline{C}), \quad \mathbf{R}_{2\pi/3}(A, B, C) = (C, A, B), \end{aligned}$$

and when $\mathbf{S}_z$ is relevant,

$$\mathbf{S}_z(A, B, C) = \pm(A, B, C).$$

The general form of vector fields commuting with these symmetries is given in [18, Chap. XIII]. When the symmetry $\mathbf{S}_z$ is irrelevant, or when it is the identity on $\mathcal{E}_0$, it is sufficient to consider the six-dimensional system truncated at order 3, of the form

$$\begin{aligned} \frac{dA}{dt} &= a\mu A + c\overline{BC} + bA|A|^2 + dA(|B|^2 + |C|^2) \\ \frac{dB}{dt} &= a\mu B + c\overline{CA} + bB|B|^2 + dB(|C|^2 + |A|^2) \\ \frac{dC}{dt} &= a\mu C + c\overline{AB} + bC|C|^2 + dC(|A|^2 + |B|^2). \end{aligned} \tag{6.26}$$

Here $\mu = \mathcal{R} - \mathcal{R}_c$, $a > 0$, and the other coefficients are all real. The coefficient b is the same as in the two-dimensional case, hence we have $b < 0$. In general the presence of quadratic terms changes drastically the stability of the steady solutions of (6.26) (see [18, Chap. XIII]). However in the present case, *a specific property of the Navier–Stokes equation implies that* $c = 0$. This comes from the fact that for any U in the domain of $\mathbf{L}$, we have

$$\langle \mathbf{R}(U), U \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $(L^2)^4$, and this scalar product arises in the computation of c, with $U = U_0$ given by (6.25).

When $B = C = 0$ we recover the Landau equation (6.17) for A, which gives the circle of steady solutions

$$a\mu + b|A|^2 = 0, \quad B = C = 0,$$

corresponding to the steady convection rolls found in the two-dimensional case. In addition, we have here the solutions obtained through the actions of $\mathbf{R}_{2\pi/3}$ and $\mathbf{S}$, which correspond to convection rolls obtained by $\pi/3$-rotations of the two-dimensional rolls above, so altogether we have *three "circles" of rolls*. In contrast to the two-dimensional case, in which these rolls are stable, here they may also be unstable. Indeed, since we have a "circle" of bifurcating solutions, one eigenvalue of the linearized operator is 0, and the other eigenvalues are now $2b|A|^2$, the same as in the two-dimensional case, and a quadruple eigenvalue $(d - b)|A|^2$. Consequently, the condition for stability of these rolls is

$$d < b < 0.$$

Another class of steady solutions of the system (6.26), with $c = 0$, is

$$A = re^{i\theta_1}, \quad B = re^{i\theta_2}, \quad C = re^{i\theta_3},$$

where $r > 0$ satisfies

$$a\mu + (b + 2d)r^2 = 0,$$

and the phases θ_j are arbitrary. For $\theta_j = 0$, this solution is invariant under the actions of $\mathbf{R}_{2\pi/3}$ and $\mathbf{S}$, and corresponds to *hexagonal convection cells* [18, Chap. XIII] (see Figure 6.3(ii)). It should be noticed by the same argument as in the two-dimensional convection, by using the periodicity and the symmetry $\mathbf{S}$, that the velocity field is tangent to the planes $x_1 = 2\pi n/k_c$ for any $n \in \mathbb{Z}$. Hence, by the D_6 rotational invariance, the velocity field is tangent to all the vertical planes deduced from this family, by rotations of angles $\pi/3$ and $2\pi/3$. This means that the fluid particles are confined in vertical triangular prisms, and a basic hexagonal prism for the pattern is formed with six of these triangular prisms. The linearized operator at these hexagonal convection cells has a triple eigenvalue 0, a simple eigenvalue $2(b + 2d)r^2$, and a double eigenvalue $2(b - d)r^2$. This latter eigenvalue implies that the hexagonal convection cells and the convection rolls cannot be both stable. In the case of "rigid-rigid" boundary conditions it is shown in [76] that $b + 2d < 0$. Actually, the result in [76] shows that hexagonal cells are stable under perturbations with hexagonal symmetry, in which case only the simple eigenvalue $2(b + 2d)r^2$ is present. We also point out that if $c \neq 0$ in system (6.26), then the phases of the steady solutions above lose one degree of freedom, and the bifurcation is two-sided. In particular, the hexagonal cells are then unstable [63, 18], but this might only apply to a different physical situation, since here $c = 0$.

In the absence of the symmetry $\mathbf{S}_z$ we need to include the fourth order terms in (6.26), in order to avoid the occurrence of a three-parameter family of hexagonal cells: Only two arbitrary phases are relevant because of the action of $\boldsymbol{\tau}_a$, and this leads to a degenerescence shown by the triple 0 eigenvalue. Adding fourth order terms (see [18] for their structure) allows us to fix $\theta_1 + \theta_2 + \theta_3 \in \{0, \pi\}$, and to obtain another, in general nonzero, simple eigenvalue decreasing by one the multiplicity of the 0 eigenvalue, for the linearized operator.

It appears that the symmetry $\mathbf{S}_z$ acts as $-\mathbb{I}$ on $\mathcal{E}_0$ in the case of the "free-free" boundary conditions, because of a factor $\sin(\pi z)$ in the components V_z and θ, and of a factor $\cos(\pi z)$ in the components V_1 and V_2 of $\widehat{U}_j(z)$, in the formula of the eigenvector ζ_j. It is a priori not automatic, but it is shown numerically that it is also the case for "rigid-rigid" boundary conditions, since for $\mathcal{R} = \mathcal{R}_c$ the components V_z and θ in $\widehat{U}_j(z)$ are invariant under the symmetry $z \mapsto 1 - z$ (see [7]). With such a symmetry, the vector field in (6.26) is odd in $(A, B, C, \overline{A}, \overline{B}, \overline{C})$, so that there are no terms of even orders. Consequently, one has to consider the fifth order terms in order to solve the degenerescence and find all steady solutions. For further details we refer to [18, Chap. XIII], where the problem is treated using the Lyapunov–Schmidt method, but the results can be adapted to the present approach. It is shown that there are four types of steady solutions: *rolls, hexagons, regular triangles*, and *patchwork quilts*, which all may be stable, depending on the coefficients, but not simultaneously. This confirms the prediction in [41], though only the first two types of solutions are usually observed.

Tridimensional Convection in an Elongated Cylindrical Domain

Finally, we briefly discuss the case of a long horizontal cylindrical container, with rectangular section in the (x_2, z)-plane, and small sides compared to the length of the cylinder along the x_1-axis. Physically, to satisfy the a priori periodicity in x_1 which we impose to the solutions, it might be convenient to take a thin ring-shaped container (a torus) having a radius large with respect to the sides of the rectangular meridian section. This problem also possesses an $O(2)$ symmetry, and it turns out to be similar to the case of two-dimensional convection [38, Vol. II]. The same approach as above can be used, showing the existence of a "circle" of *stable convection rolls*, bifurcating for $\mathcal{R} > \mathcal{R}_c$, which are periodic in x_1, the cells being parallel to the x_2-axis.

Second Bifurcation We are interested here in the next bifurcation, when $\mathcal{R}$ crosses a second critical value $\mathcal{R}_2$, at which the stable convection rolls for $\mathcal{R} > \mathcal{R}_c$ become unstable.

The "circle" of convection rolls is given by $\boldsymbol{\tau}_a U_*$, $a \in \mathbb{R}$, where U_* is a symmetric solution, $\mathbf{S}U_* = U_*$. Notice that there are two such symmetric solutions on the "circle," and that all these solutions are of class $\mathcal{C}^\infty$. The generator of the group $(\boldsymbol{\tau}_a)_{a\in\mathbb{R}}$ is the derivative $\partial_{x_1} \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$, and then $\partial_{x_1} U_*$, the Goldstone mode, satisfies

$$\partial_{x_1} U_* \in \mathcal{Z}, \quad (\mathbf{L} + D_U \mathbf{R}(U_*))(\partial_{x_1} U_*) = 0, \quad \mathbf{S}(\partial_{x_1} U_*) = -\partial_{x_1} U_*.$$

In particular, this shows that the operator $\mathbf{L}+D_U\mathbf{R}(U_*)$ has an eigenvalue 0 with eigenvector $\partial_{x_1} U_*$. It turns out, that experimental evidence suggests that this eigenvalue is actually algebraically double and geometrically simple when $\mathcal{R} = \mathcal{R}_2$. Indeed, for $\mathcal{R}$ close to $\mathcal{R}_2$ there are bifurcating solutions which are slow traveling waves, and, as we shall see below, correspond to the situation in which there is a generalized antisymmetric eigenvector ξ_0, such that

$$(\mathbf{L} + D_u \mathbf{R}(U_*))\xi_0 = \partial_{x_1} U_*, \quad \mathbf{S}\xi_0 = -\xi_0,$$

(see also [8, p. 102], for an analogue for the Couette–Taylor problem).

Following the method of construction of center manifolds near a line of equilibria in Section 4.3.4, we consider the new coordinates (α, v) defined through

$$U = \boldsymbol{\tau}_\alpha(U_* + v), \quad \langle v, \partial_{x_1} U_* \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $(L^2)^4$. Then the linear operator $\mathbf{L}'$ defined in (4.29), acting on v, which commutes with $\mathbf{S}$ due to the choice of U_*, has a simple eigenvalue crossing the imaginary axis through 0, when $\mathcal{R}$ crosses $\mathcal{R}_2$. Applying the center manifold Theorems 4.34 and 4.29, we conclude that a *pitchfork bifurcation* occurs in the equation for v when $\mathcal{R} = \mathcal{R}_2$ (see also the general study of the ten possible solutions generically bifurcating from a one-dimensional periodic pattern in [11]). Since $\alpha(t)$ has a small constant derivative, the bifurcating solutions are *traveling waves with speeds close to* 0, which arise in pairs exchanged by the symmetry $\mathbf{S}$, i.e., traveling in opposite directions. This type of flow is indeed observed in experiments [4].

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