Introduction to Hydrodynamic Instabilities

François Charru
Institut de Mécanique des Fluides de Toulouse
CNRS – Univ. Toulouse

École d'été sur les Instabilités et Bifurcations en Mécanique
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Contents

References

This presentation is based on the book

Hydrodynamic Instabilities (F. Charru 2011, Cambridge Univ. Press)

where the references of all the pictures can be found. Other references:

Textbooks on Fluid Mechanics

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- Landau L. & Lifschitz E. 1987 *Fluid Mechanics*, Butterworth-Heinemann.
- Tritton D.J. 1988 *Physical Fluid Dynamics*, Clarendon Press.

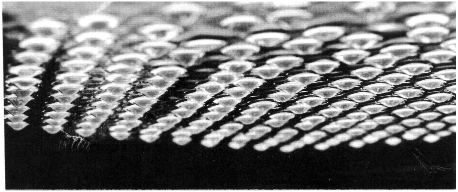
Specialized Textbooks

- Drazin P.G. 2002 *Introduction to Hydrodynamic Stability*, Cambridge UP.
- Huerre P. & Rossi M. 1998 *Hydrodynamic Instabilities in Open Flows*. Eds Godrèche C. & Manneville P., Cambridge UP.
- Manneville P. 1990 *Dissipative Structures and Weak Turbulence*, Academic Press.
- Schmid P.J. & Henningson D.S. *Stability and Transition in Shear Flows*, Springer-Verlag.

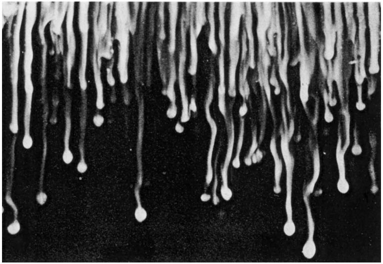
1. Instabilities of fluids at rest

Gravity-driven Rayleigh-Taylor instability (1)

Pending drops under a suspended liquid film



Descending fingers of salt water into fresh water



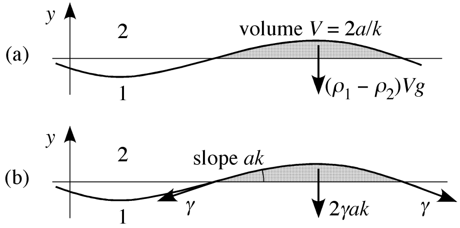
Gravity-driven Rayleigh-Taylor instability (2)

Analysis with viscosity and bounding walls neglected.

Base state:

- fluids at rest with horizontal interface,
- hydrostatic pressure distribution.

Perturbed flow:

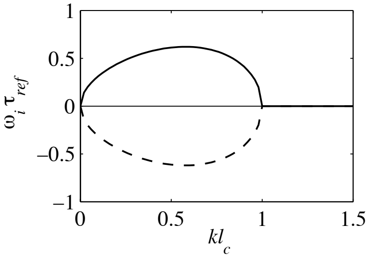


Gravity-driven Rayleigh-Taylor instability (3)

Linearized perturbation equations and perturbations $\propto \mathrm{e}^{\mathrm{i}(kx-\omega t)}$

$$\rightarrow \text{Dispersion relation} \quad \omega^2 = \frac{(\rho_1 - \rho_2)gk + k^3\gamma}{\rho_1 + \rho_2}$$

$\rightarrow$ Instability (complex ω) when $\rho_1 < \rho_2$, with growth rate:
(l_c capillary length, τ_{ref} capillary time)



$\rightarrow$ Long-wave instability

Instabilities related to Rayleigh-Taylor

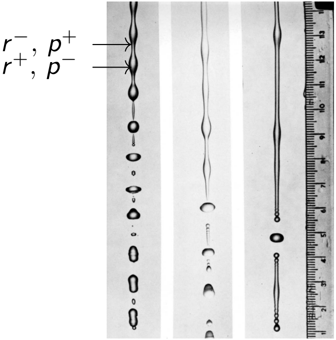
Inertial instability of accelerated flows (Taylor 1950)

Gravitational instability in astrophysics (Jeans 1902)

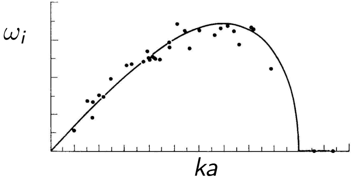
$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0.$$

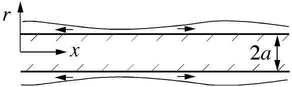
Capillary Rayleigh-Plateau instability

jet of water

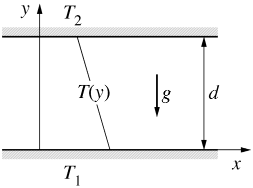


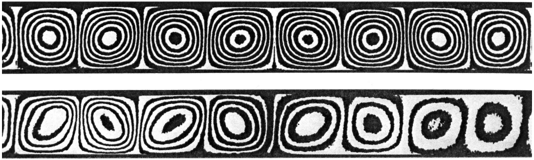
drops on a spider web



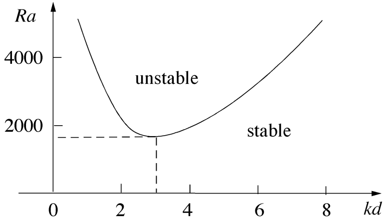


Buoyancy-driven Rayleigh-Bénard instability





Linear stability analysis → dispersion curve

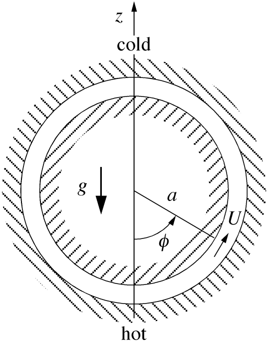


Bifurcation parameter:
Rayleigh number

$$\mathrm{Ra} = \frac{\alpha_\mathrm{p} g (T_1 - T_2) d^3}{\nu\kappa}$$

A toy-model: convection in an annulus (1)

(Welander 1967)



Base state: fluid at rest with temperature

$$\overline{T} = T_0 - T_1 \frac{z}{a} = T_0 + T_1 \cos\phi.$$

Momentum conservation:

$$\frac{\partial U}{\partial t} = -\frac{1}{\rho a}\frac{\partial p}{\partial \phi} + \alpha g (T - \overline{T}) \sin\phi - \gamma U.$$

Energy conservation:

$$\frac{\partial T}{\partial t} + \frac{U}{a}\frac{\partial T}{\partial \phi} = k(T - \overline{T}).$$

Temperature sought for as

$$T(t,\phi) = \overline{T} + T_A(t)\sin\phi - T_B(t)\cos\phi,$$

A toy-model: convection in an annulus (2)

The change of scales

$$X \propto U, \qquad Y \propto T_A, \qquad Z \propto T_B, \qquad \tau \propto t,$$

then provides the Lorenz system (1963)

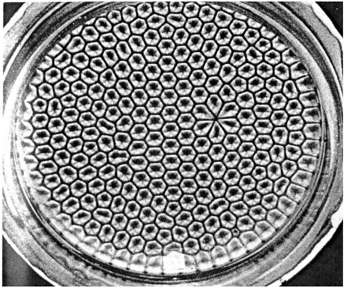
$$\begin{aligned} \partial_\tau X &= -PX + PY \\ \partial_\tau Y &= -Y - XZ + RX \\ \partial_\tau Z &= -Z + XY \end{aligned}$$

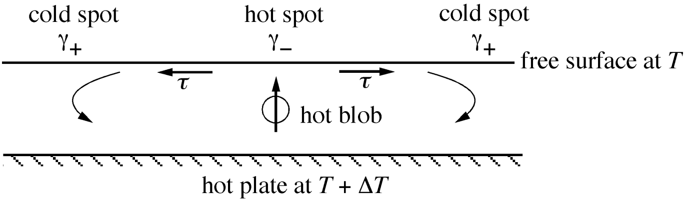
$$\text{where} \qquad P = k/\gamma, \qquad R = \alpha g T_1 / 2\gamma k a.$$

Stability analysis of the fixed point $(0, 0, 0)$ (fluid at rest)
$\rightarrow$ Supercritical pitchfork bifurcation at $R_{c1} = 1$ (convection)

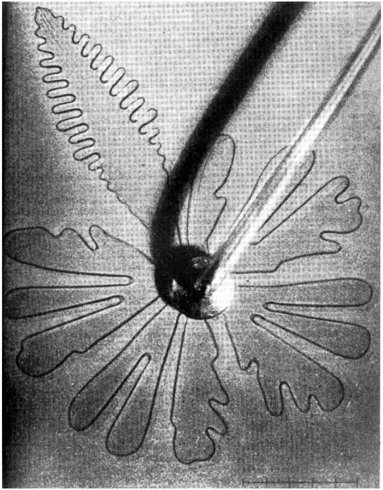
Chaotic behavior beyond $R_{c2}(P) > R_{c1}$ via a subcritical Hopf bifurcation (Lorenz strange attractor).

Thermocapillary Bénard-Marangoni instability





Saffman-Taylor instability of fronts between viscous fluids



2. Stability of open flows: basic ideas

Forced flow: canonic forcings

Consider the (1D) linearized evolution equation for $u(x,t)$

$$L\,u(x,t) = S(x,t)$$

L: differential linear operator involving x- and t-derivatives
$S(x,t)$: forcing.

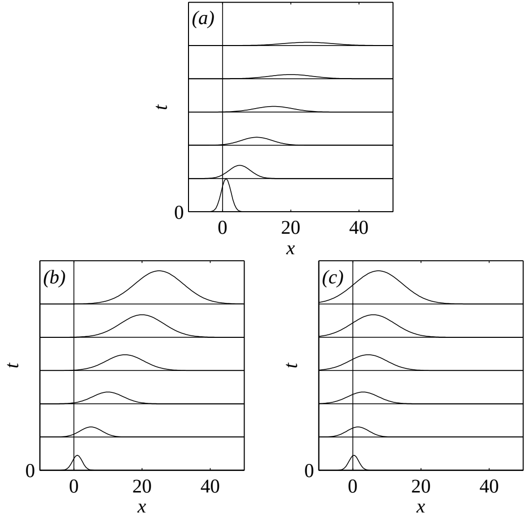
Three types of elementary forcing functions of special importance:

$$S(x,t) = F(x)\delta(t) \qquad \textit{(initial value problem)}$$
$$S(x,t) = \delta(x)\delta(t) \qquad \textit{(impulse response problem)}$$
$$S(x,t) = \delta(x)H(t)\mathrm{e}^{-\mathrm{i}\omega t} \qquad \textit{(periodic forcing problem)}$$

where δ and H are the Dirac and Heaviside functions.

Impulse response – Definitions

Spatiotemporal evolution of a disturbance localized at $x = 0$ at $t = 0$

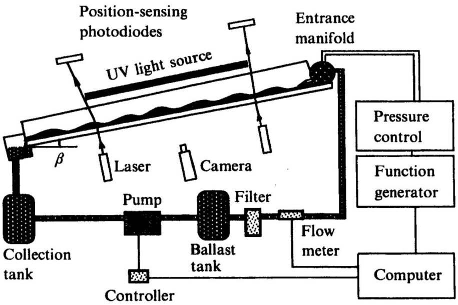


(a): Linearly stable flow

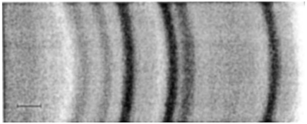
(b): Linearly unstable flow, convective instability

(c): Linearly unstable flow, absolute instability

Illustration: waves on a falling film (1)



Natural waves



Forced waves, 5.5 Hz

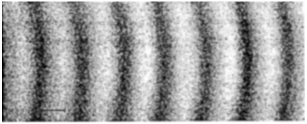
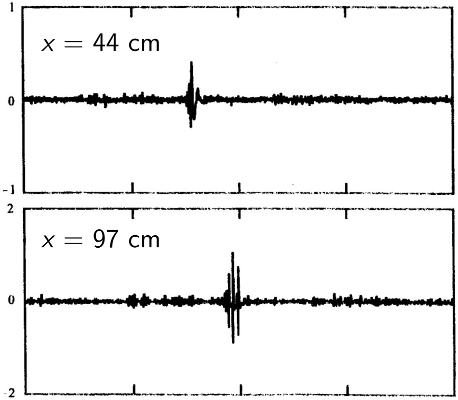


Illustration: waves on a falling film (2)

A perturbation generated at $x = t = 0$ amplifies while it is convected downstream:



Stability criteria

It can be shown that:

- A necessary and sufficient condition for stability is that the growth rates of all the modes with real wavenumber k are negative (temporal stability)
- The criterion for absolute instability is that there exists some wavenumber k_0 with zero group velocity and positive growth rate.

A convective instability amplifies any unstable perturbation, and advects it downstream ("noise amplifier")

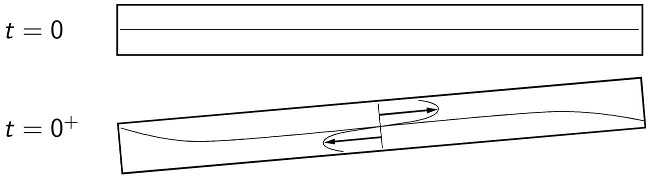
An absolutely unstable flow responds selectively to the perturbation with zero group velocity: it behaves like an oscillator with its own natural frequency.

3. Inviscid instability of parallel flows

- Large Reynolds number flows (negligible viscous effects)
- Far from solid boundaries

Illustration 1: tilted channel

(Reynolds 1883, Thorpe 1969)



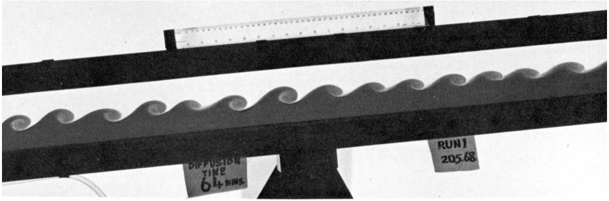
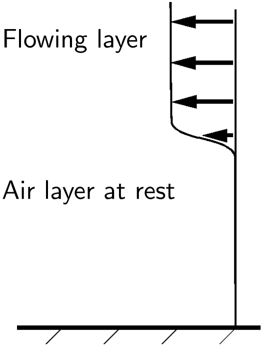


Illustration 2: wind in a stratified atmosphere



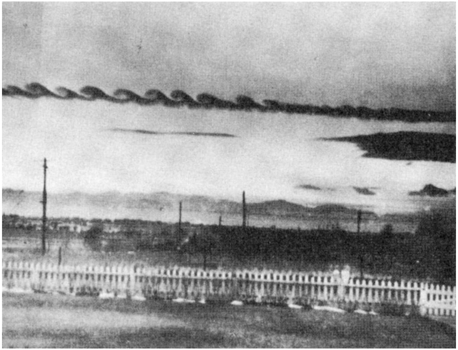
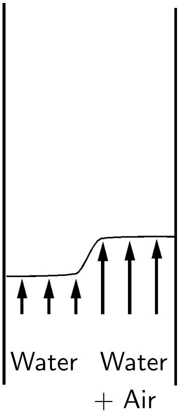


Illustration 3: rising mixing layer



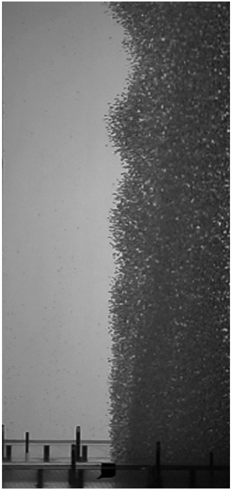
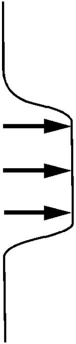
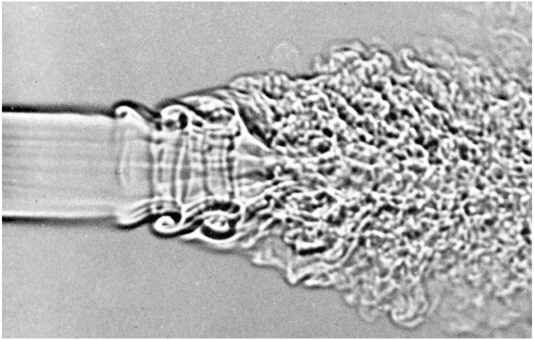


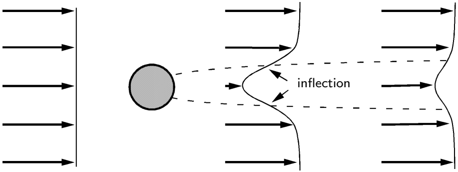
Illustration 4: jet

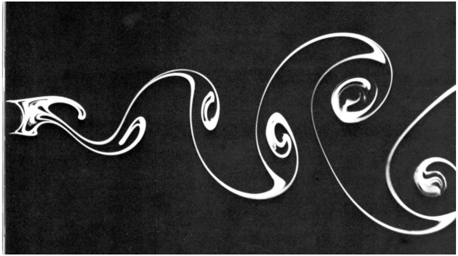




Jet of carbone dioxyde 6 mm in diameter issuing into air at a speed of $40\,\mathrm{m\,s^{-1}}$ (Re = 30 000).

Illustration 5: wake





Wake of a cylinder in water flowing at $1.4\,\mathrm{cm\,s^{-1}}$ ($\mathrm{Re} = 140$).

General results – Base flow

Ignoring viscous effects, and with unit scales L, V and ρV^2, the governing equations are the Euler equations

$$\operatorname{div} \mathbf{U} = 0,$$
$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \mathbf{grad})\mathbf{U} = -\mathbf{grad}\, P.$$

These equations have the family of base solutions

$$\overline{\mathbf{U}}(\mathbf{x}, t) = \overline{U}(y)\mathbf{e}_x, \qquad \overline{P}(\mathbf{x}, t) = \overline{P},$$

corresponding to parallel flow.

General results – Linearized stability problem

Linearized equations for the perturbed base flow $\overline{\mathbf{U}} + \mathbf{u}$, $\overline{P} + p$

$$\begin{aligned} \operatorname{div} \mathbf{u} &= 0, \\ (\partial_t + \overline{U}\partial_x)\mathbf{u} + v\partial_y \overline{U}\, \mathbf{e}_x &= -\mathbf{grad}\, p. \end{aligned}$$

Thanks to the translational invariance in t, x and z, the solution can be sought in the form of normal modes such as

$$u(\mathbf{x}, t) = \hat{u}(y)\mathrm{e}^{\mathrm{i}(k_x x + k_z z - \omega t)} + \mathrm{c.c.},$$

whose amplitudes $\hat{u}(y), \ldots$ satisfy the homogeneous system:

$$\begin{aligned} \mathrm{i}k_x\hat{u} + \partial_y\hat{v} + \mathrm{i}k_z\hat{w} &= 0, \\ \mathrm{i}(k_x\overline{U} - \omega)\hat{u} + \partial_y\overline{U}\hat{v} &= -\mathrm{i}k_x\hat{p}, \\ \mathrm{i}(k_x\overline{U} - \omega)\hat{v} &= -\partial_y\hat{p}, \\ \mathrm{i}(k_x\overline{U} - \omega)\hat{w} &= -\mathrm{i}k_z\hat{p}. \end{aligned}$$

with the conditions that the perturbations fall off for $y \to \pm\infty$ or that $\hat{v}(y_1) = \hat{v}(y_2) = 0$ at impermeable walls.

General results – Dispersion relation

The above system can formally be written as the generalized eigenvalue problem

$$L\phi = \omega M\phi,$$

where $\phi = (\hat{u}, \hat{v}, \hat{w}, \hat{p})$ and L, M linear differential operators.

This problem has a nonzero solution ϕ only if the operator $L - \omega M$ is noninvertible, *i.e.*, if for a given wave number the frequency ω is an eigenvalue. This condition can be written formally as

$$D(\mathbf{k}, \omega) = 0,$$

which is the dispersion relation of perturbations of infinitesimal amplitude.

General results – Reduction to a 2D problem

Using the Squire transformation

$$\widetilde{k}^2 = k_x^2 + k_z^2, \qquad \widetilde{\omega} = (\widetilde{k}/k_x)\,\omega,$$
$$\widetilde{k}\widetilde{u} = k_x\hat{u} + k_z\hat{w}, \qquad \widetilde{v} = \hat{v}, \qquad \widetilde{p} = (\widetilde{k}/k_x)\,\hat{p}$$

the governing equations become, with $\widetilde{c} = c = \omega/k_x$

$$\begin{aligned} \mathrm{i}\widetilde{k}\widetilde{u} + \partial_y\widetilde{v} &= 0, \\ \mathrm{i}\widetilde{k}(\overline{U} - \widetilde{c})\widetilde{u} + \partial_y\overline{U}\,\widetilde{v} &= -\mathrm{i}\widetilde{k}\widetilde{p}, \\ \mathrm{i}\widetilde{k}(\overline{U} - \widetilde{c})\widetilde{v} &= -\partial_y\widetilde{p}, \end{aligned}$$

General results – Squire theorem

Knowing the dispersion relation of the two-dimensional system

$$\widetilde{D}(\widetilde{k},\widetilde{\omega}) = 0,$$

the dispersion relation for three-dimensional perturbations can be obtained by means of the Squire transformation:

$$D(\mathbf{k},\omega) = \widetilde{D}\left(\sqrt{k_x^2+k_z^2}, \frac{\sqrt{k_x^2+k_z^2}}{k_x}\omega\right) = 0.$$

→ **Squire's theorem**. For any three-dimensional unstable mode $(\mathbf{k},\omega)$ of temporal growth rate ω_i there is an associated two-dimensional mode $(\widetilde{k},\widetilde{\omega})$ of temporal growth rate $\widetilde{\omega}_i = \omega_i\sqrt{k_x^2+k_z^2}/k_x$, which is more unstable since $\widetilde{\omega}_i > \omega_i$.

Therefore when the problem is to determine an instability condition it is sufficient to consider only two-dimensional perturbations.

The Rayleigh equation and inflection point theorem

Introducing the stream function, the 2D stability problem reduces to the Rayleigh equation

$$(\overline{U} - c)(\partial_{yy}\hat{\psi} - k^2\hat{\psi}) - \partial_{yy}\overline{U}\,\hat{\psi} = 0$$

Thus, if $\hat{\psi}$ is eigenfunction with eigenvalue c, then so are $\hat{\psi}^*$ and c^*: stability implies real c ($c_i = 0$, i.e. neutral stability).

The Rayleigh theorem. An inflection point in the velocity profile $\overline{U}(y)$ is a necessary (but not sufficient) condition for instability.

Assume that the flow is unstable ($c_i \neq 0$). Divide the Rayleigh equation by $(\overline{U} - c)$, multiply by $\hat{\psi}^*$, integrate by parts between the walls, with $\hat{\psi}(y_1) = \hat{\psi}(y_2) = 0$, the imaginary part of the result is

$$c_i \int_{y_1}^{y_2} \frac{\partial_{yy}\overline{U}}{|\overline{U} - c|^2} |\hat{\psi}|^2 dy = 0.$$

Since $c_i \neq 0$ by assumption, $\partial_{yy}\overline{U}$ must change sign.

Jump conditions for piecewise-linear velocity profile

The eigenfunctions of the perturbations are exponentials within each layer, and only need to be matched at the discontinuities.
Let $y = y_0 + \eta(x,t)$ be the perturbed position of a discontinuity at $y = y_0$, and $\mathbf{n}$ be the normal. The normal velocity of the fluid must be continuous and equal to the velocity $\mathbf{w} \cdot \mathbf{n}$ of the interface:

$$(\mathbf{U}_+ \cdot \mathbf{n})(y_0 + \eta) = (\mathbf{U}_- \cdot \mathbf{n})(y_0 + \eta) = \mathbf{w} \cdot \mathbf{n}.$$

Linearizing at $y = y_0$, with $\mathbf{n} = (-\partial_x \eta, 1)$ and $\mathbf{w} \cdot \mathbf{n} = -\partial_t \eta$, introducing the normal modes and eliminating $\hat{\eta}$ gives:

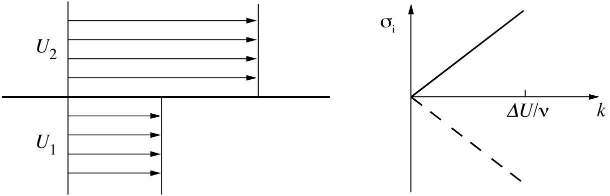
$$\Delta \left(\frac{\hat{\psi}}{\overline{U} - c} \right) = 0, \qquad \text{where} \quad \Delta[X] = X_+(y_0) - X_-(y_0).$$

The continuity of pressure gives similarly

$$\Delta[(\overline{U} - c)\partial_y \hat{\psi} - \partial_y \overline{U}\, \hat{\psi}] = 0.$$

$\rightarrow$ Complete determination of the eigenfunctions.

Kelvin-Helmholtz instability of a vortex sheet



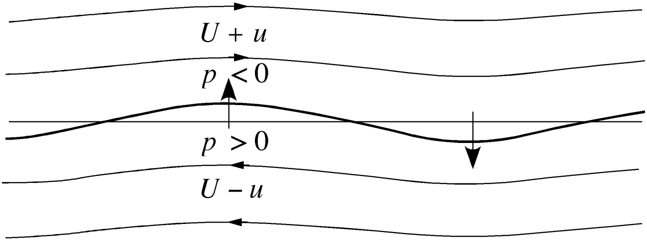
The solution of the Rayleigh equation, $\hat{\psi}_j = A_j \mathrm{e}^{-ky} + B_j \mathrm{e}^{ky}$, $j = 1, 2$, the fall-off of the perturbations at infinity ($A_1 = 0$ and $B_2 = 0$ for $k > 0$), and the jump conditions at the interface gives

$$\begin{aligned}(U_1 - c)A_2 - (U_2 - c)B_1 &= 0\\ (U_2 - c)A_2 + (U_1 - c)B_1 &= 0\end{aligned}$$

which has a nontrivial solution only when (dispersion relation) $(U_1 - c)^2 + (U_2 - c)^2 = 0$,

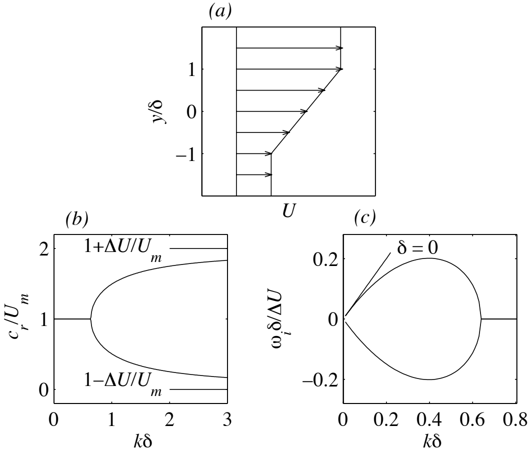
$$i.e. \quad c = \frac{\omega}{k} = U_{\mathrm{av}} \pm \mathrm{i}\Delta U, \quad with \quad 2U_{\mathrm{av}} = U_1 + U_2, \quad 2\Delta U = U_1 - U_2.$$

Mechanism of the Kelvin-Helmholtz instability



"Bernoulli effect"

Kelvin-Helmholtz with vorticity layer of finite thickness 2δ



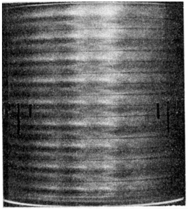
$\rightarrow$ Stable short waves, and long-wave instability with $\omega_{i,max} \approx 0.2U/\delta$

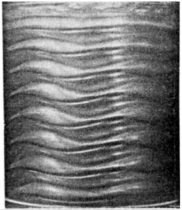
Inviscid analysis valid whenever $\delta/\Delta U \ll \delta^2/\nu$, i.e. $Re \gg 1$.
Viscous effects decrease ω_i and k_{cutoff} (also increase δ).

Couette-Taylor centrifugal instability

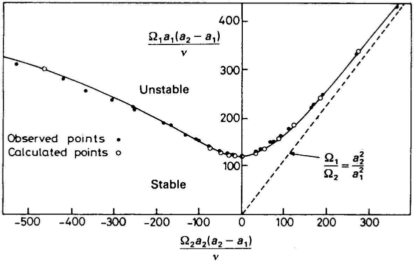
The Couette flow between two coaxial cylinders may be unstable due to the centrifugal force.

primary vortices (co-rotation)





secondary undulated vortices



Landmark experiments by G. I. Taylor (1923)
1: inner
2: outer
a_2/a_1 fixed

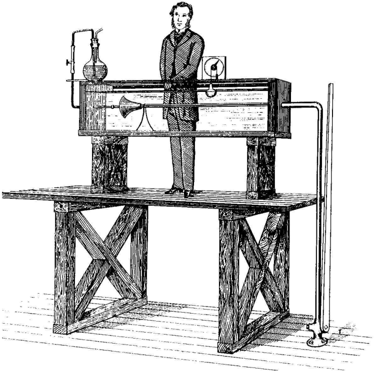
Rayleigh (1916): a stable stratification of centrifugal force satisfies $\Omega_1 a_1^1 < \Omega_2 a_2^2$.

4. Viscous instability of parallel flows

- Boundary layers and Poiseuille flow have no inflection point
- However, experiments show that that they may be unstable...

Illustration 1: Poiseuille flow in a tube

(Reynolds 1883)



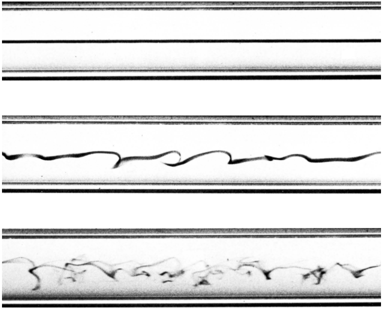
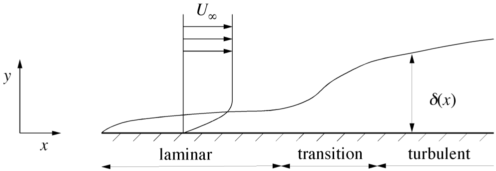
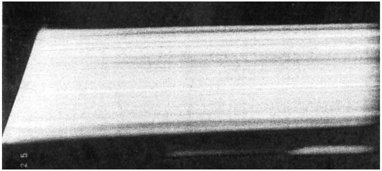
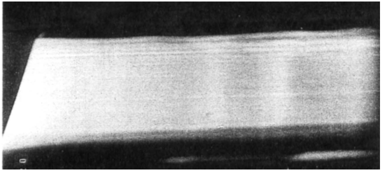


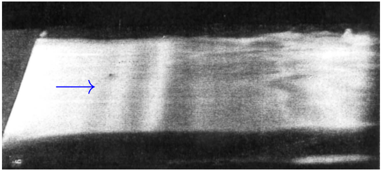
Illustration 2: Boundary layer







Tollmien-Schlichting waves



General results – 2nd Squire's theorem

- The analysis goes along the same lines as for inviscid flow, with the viscous diffusion term

$$\frac{1}{Re}\Delta \mathbf{U}, \qquad Re = \frac{UL}{\nu}$$

- The generalized eigenvalue problem has nontrivial solution when the operator is singular, i.e. $D(\mathbf{k}, \omega, Re) = 0$.
- Squire's theorem. For any unstable oblique mode $(\mathbf{k}, \omega)$ of temporal growth rate ω_i for Reynolds number Re it is possible to associate a two-dimensional mode $(\widetilde{k}, \widetilde{\omega})$ of temporal growth rate $\widetilde{\omega}_i = \omega_i \sqrt{k_x^2 + k_z^2}/k_x$, higher than ω_i, at a Reynolds number $\widetilde{\mathrm{Re}} = \mathrm{Re}\, k_x/\sqrt{k_x^2 + k_z^2}$, lower than Re.
  Corollary. If there exists some $\mathrm{Re_c}$ above which a flow is unstable, the destabilizing normal mode for $\mathrm{Re} = \mathrm{Re_c}$ is 2D.

The Orr-Sommerfeld equation

(Orr 1907, Sommerfeld 1908)

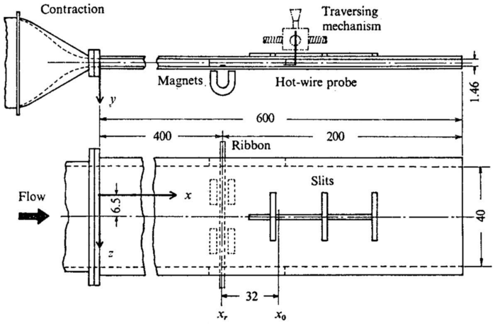
For plane flow, 2D disturbances obey the The Orr-Sommerfeld equation (Rayleigh equation + viscous term):

$$(\overline{U} - c)(\partial_{yy} - k^2)\hat{\psi} - \partial_{yy}\overline{U}\,\hat{\psi} = \frac{1}{\mathrm{i}k\mathrm{Re}}(\partial_{yy} - k^2)^2\hat{\psi}.$$

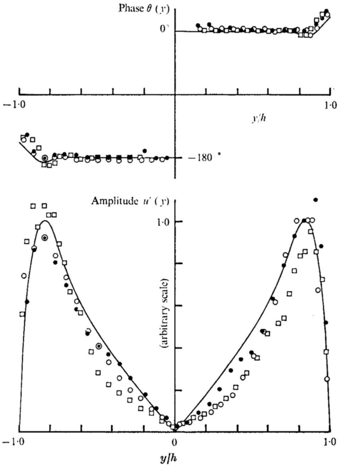
- No exact solution except for linear velocity profile (integrals of Airy functions)
- Difficult to solve for high Re, especially near the critical layer (where $c = \overline{U}$) and near walls or interfaces
- May be solved analytically using perturbation methods (small or large k, small or large Re)
- May be solved numerically using shooting or spectral methods

Plane Poiseuille flow (1)

Experiment (Nishioka *et al.* 1075):
a ribbon excites Tollmien-Schlichting waves.



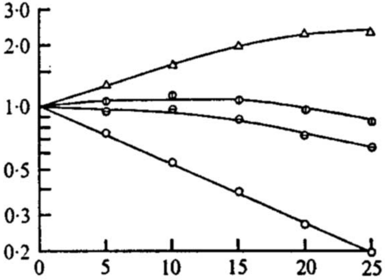
Eigenfunctions:

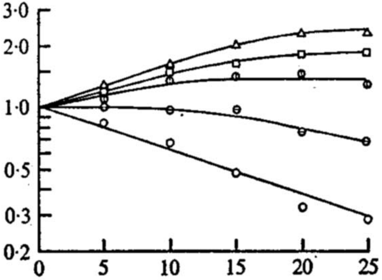


Plane Poiseuille flow (2)

Spatial evolution of the forced disturbances ...

for increasing Re = 4000...8000 ... for varying frequency



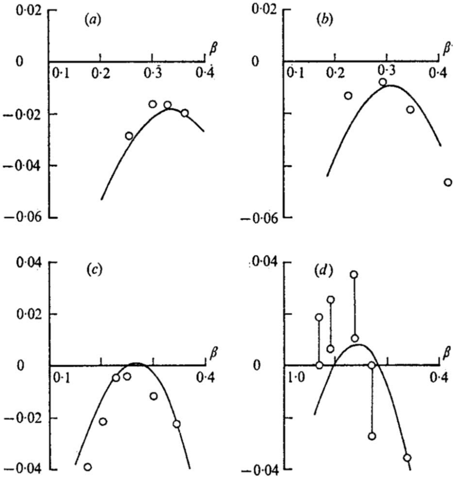


The spatial and temporal growth rates are related through the group velocity:

$$\omega_i^T = -c_g k_i^S \qquad \text{(Gaster 1962)}$$

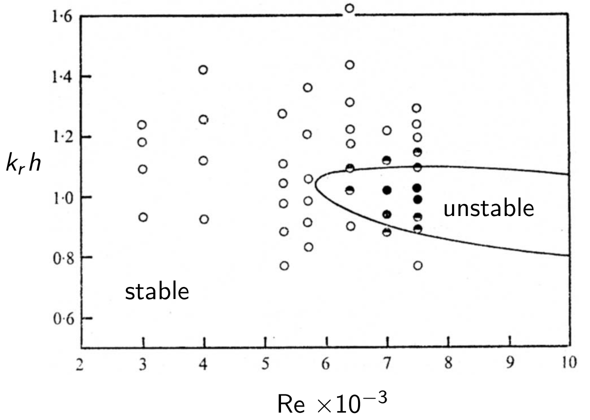
Plane Poiseuille flow (3)

Spatial growth rates versus frequency $\beta = 2\pi f U/h$
for increasing Re = 3000...7000 and comparison with calculations



Plane Poiseuille flow (4)

Stability diagram

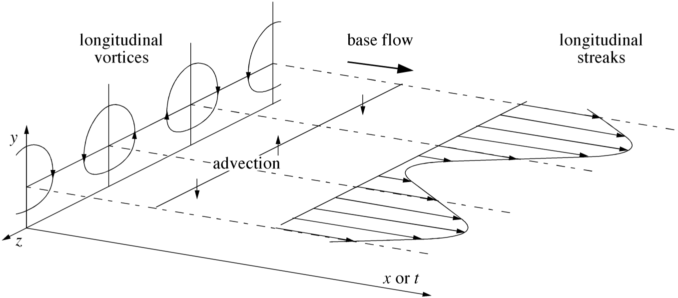


Transient growth

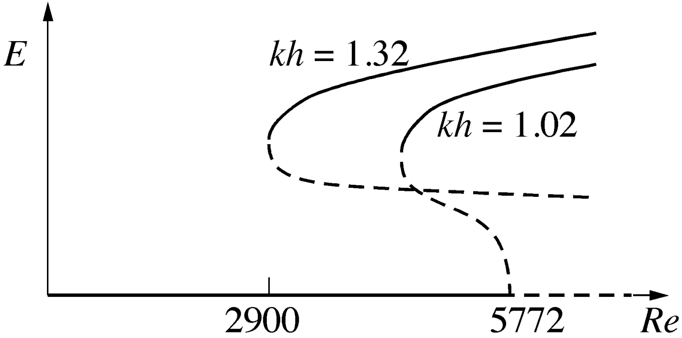
When the disturbance is not well controlled (amplitude $\gtrsim 1\%$), instability occurs well below $\mathrm{Re} = 5772$.

An explanation relies on the non-normality of the Orr-Sommerfeld operator and the associated Orr equation for the y-vorticity, which implies transient growth of the superposition of stable eigenmodes, and may trigger nonlinear behaviour.

Longitudinal vortices give rise to the strongest transient growth (optimal perturbation).



Plane Poiseuille flow: tentative bifurcation diagram

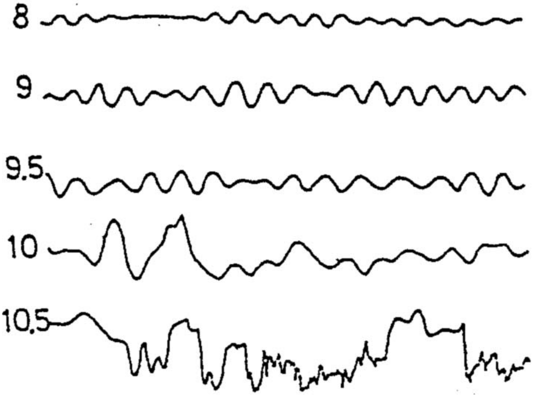


Poiseuille flow in a pipe

... is linearly stable!

Difficulties begin...

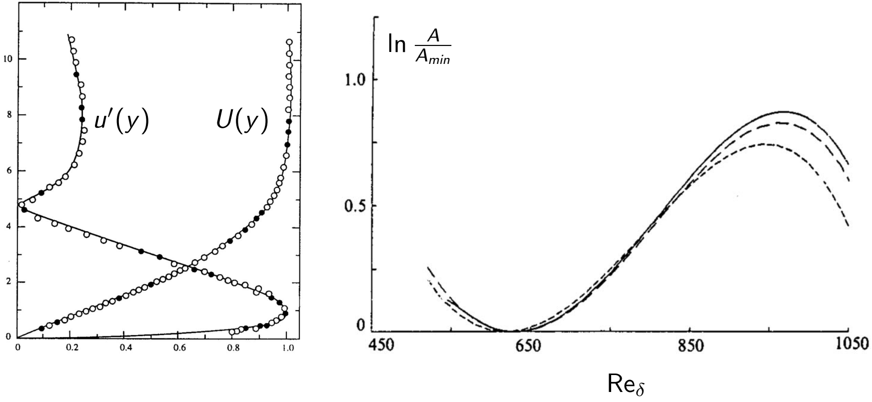
Boundary layer on a flat plate (1)



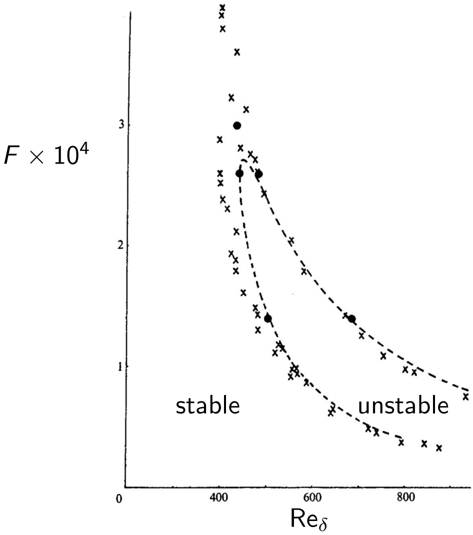
Velocity fluctuations of a forced Tollmien–Schlichting wave, measured at different positions (in feet) downstream from the leading edge, for upstream velocity $U_\infty = 36.6 \text{ m s}^{-1}$ (Schubauer & Skramstad 1947).

Boundary layer on a flat plate (2)

Although the flow is not strictly parallel ($\delta(x)$ increases),
local analysis is possible with $U(x,y)$, and x treated as a parameter.



Boundary layer on a flat plate (3)



Marginal stability: (- -) nonparallel theory, (×) measurements, (•) DNS.

5. Nonlinear dynamics with few degrees of freedom

- What happens beyond the exponential growth, when nonlinear terms are no longer negligible?
- No general theory
- 'Weakly nonlinear analysis' particularly important owing to its fairly general nature based on perturbation methods.

Beyond the exponential growth: the Landau equation

According to the linear stability theory, a perturbation of a base flow can be expressed as a sum of uncoupled eigenmodes:

$$u(\mathbf{x}, t) = \frac{1}{2}\left(A(t)f(\mathbf{x}) + A^*(t)f^*(\mathbf{x})\right)$$

$f(\mathbf{x})$: spatial structure of the mode, $A(t)$ its time evolution.

Basic idea (Landau 1944): $A(t)$ grows exponentially as the solution of

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \sigma A, \qquad \sigma \quad \text{temporal growth rate}$$

This equation can be viewed as a Taylor series expansion of $\mathrm{d}A/\mathrm{d}t$ in powers of A, truncated at first order. For problems invariant under time translation, the equation for A must be invariant under the rotations $A \to A\mathrm{e}^{\mathrm{i}\phi}$. The lowest order term satisfying this condition is $|A|^2 A$. Hence the Landau equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \sigma A - \kappa |A|^2 A.$$

The unkown coefficient κ can be determined by means of a perturbation expansion for small amplitude.

Calculations of the Landau constants

The Landau constant has been calculated for the major instabilities, through the expansion of the governing equations in power series of the amplitude.

- Rayleigh-Bénard problem: κ_r is positive, corresponding to supercritical pitchfork bifurcation (Gor'kov 1957, Malkus & Veronis 1958)
- Taylor-Couette flow: same conclusion (see Chossat & Iooss 1994)
- Plane Poiseuille flow: the instability at $\mathrm{Re} = 5772$ is subcritical: no saturation by the cubic term (Stuart 1958, Watson 1960). More work is needed...

The expansion procedure is illustrated below with nonlinear oscillators.

Van der Pol oscillator: saturation of the amplitude (1927)

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - (2\epsilon\mu - u^2)\frac{\mathrm{d}u}{\mathrm{d}t} + \omega_0^2 u = 0, \qquad \mu = \mathcal{O}(1), \quad \epsilon \ll 1.$$

The fixed point $(u, du/dt) = (0, 0)$ is a stable focus for $\mu < 0$, unstable for $\mu > 0$.

- Growth rate $\epsilon\mu \ll \omega_0$: slow variation of the amplitude expected
- For $\mu > 0$, saturation expected for $u \sim \epsilon^{1/2}$.

Hence, $u(t)$ sought for as (multiple scale expansion)

$$u(t) = \epsilon^{1/2}\widetilde{u}(t, T), \qquad T = \epsilon t, \quad \widetilde{u} = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots$$

Van der Pol: solution at order ϵ^0

At the dominant order, the linear problem to solve is

$$Lu_0 = 0 \qquad \text{with} \quad L = \frac{\partial^2}{\partial \tau^2} + \omega_0^2$$

with solution

$$u_0 = \frac{1}{2}\left(A(T)\mathrm{e}^{\mathrm{i}\omega_0\tau} + A(T)^*\mathrm{e}^{-\mathrm{i}\omega_0\tau}\right).$$

Van der Pol: solution at order ϵ^1

At the next order, the linear nonhomogeneous problem to solve is

$$Lu_1 = -2\frac{\partial^2 u_0}{\partial\tau\partial T} + (2\mu - u_0^2)\frac{\partial u_0}{\partial\tau}$$

with r.h.s. known from the previous step, so that:

$$Lu_1 = \mathrm{i}\omega_0\left(\mu A - \frac{\mathrm{d}A}{\mathrm{d}T}\right)\mathrm{e}^{\mathrm{i}\omega_0\tau} - \frac{\mathrm{i}\omega_0}{8}\left(|A|^2 A\mathrm{e}^{\mathrm{i}\omega_0\tau} + A^3\mathrm{e}^{3\mathrm{i}\omega_0\tau}\right) + \mathrm{c.c.},$$

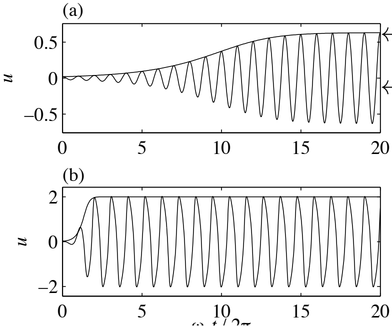
Cancellation of the resonant forcing (solvability condition) leads to

$$\frac{\mathrm{d}A}{\mathrm{d}T} = \mu A - \kappa|A|^2 A, \qquad \kappa = \frac{1}{8} \qquad \text{(Landau equation)}$$

$$A = a(T)\mathrm{e}^{\mathrm{i}\phi(T)} \quad \longrightarrow \quad \frac{\mathrm{d}a}{\mathrm{d}T} = \mu a - \kappa a^3, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}T} = 0.$$

- Supercritical Hopf bifurcation at $\mu = 0$
- The nonlinearity saturates the amplitude.

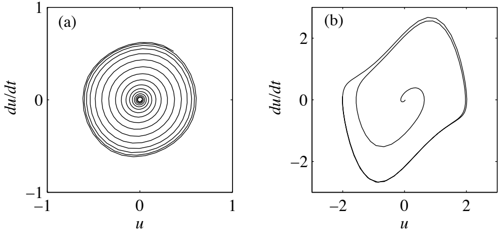
Van der Pol: asymptotic vs. numerical solutions



← Landau equation

← Numerical solution

Farther from threshold

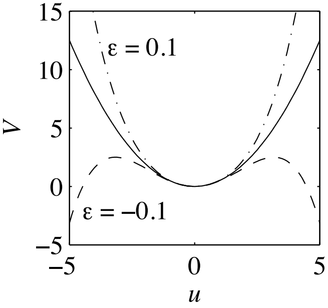


Duffing oscillator: frequency correction

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + u + \epsilon u^3 = 0, \qquad \mu = \mathcal{O}(1), \quad \epsilon \ll 1.$$

Can be written:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = -V'(u), \qquad V(u) = \frac{u^2}{2} + \epsilon \frac{u^4}{4},$$



Duffing: multiple scale analysis

$$\text{Expand as before} \qquad u(t) = u_0(\tau, T) + \epsilon u_1(\tau, T) + \ldots$$

$$\longrightarrow \qquad Lu_0 = 0 \qquad \text{with} \quad L = \frac{\partial^2}{\partial \tau^2} + 1,$$

$$Lu_1 = -2\frac{\partial^2 u_0}{\partial \tau \partial T} - u_0^3.$$

The solvability condition at order ϵ gives the Landau equation

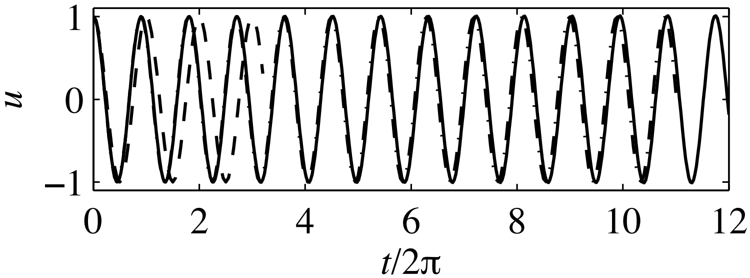
$$\frac{\mathrm{d}A}{\mathrm{d}T} = \frac{3\mathrm{i}}{8}|A|^2 A \qquad \text{(no linear term)}$$

$$A = a(T)\mathrm{e}^{\mathrm{i}\phi(T)} \quad \longrightarrow \quad \frac{\mathrm{d}a}{\mathrm{d}T} = 0, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}T} = \frac{3}{8}a^2.$$

Hence the final solution:

$$u(t) = a_0 \cos(\omega t + \phi_0) + \mathcal{O}(\epsilon), \qquad \omega = 1 + \frac{3}{8}\epsilon a_0^2 + \mathcal{O}(\epsilon^2).$$

Duffing: asymptotic vs. numerical solutions



dashed: $O(\epsilon^0)$ solution

dashed-dotted: $O(\epsilon^1)$ solution

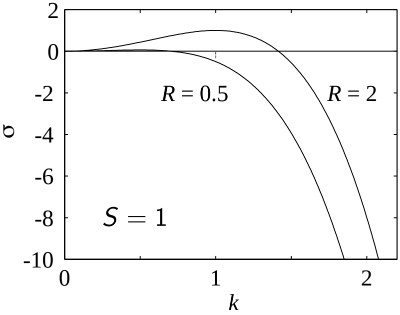
plain curve: numerical solution

Derivation of the Landau equation from the Kuramoto-Sivashinsky (KS) equation

$$\partial_t u + 2V\, u\partial_x u + R\,\partial_{xx} u + S\,\partial_{xxxx} u = 0.$$

Normal modes $\propto \mathrm{e}^{\sigma t + \mathrm{i}(\omega t - kx)}$ $\rightarrow$ dispersion relation:

$$\sigma = Rk^2 - Sk^4, \qquad \omega = 0$$



KS: amplitude expansion

Search for periodic solutions with wavelength $L = 2\pi/k_1$.
Rescale u, x and t so that $k_1 = 1$, $S = 1$, and expand in Fourier series

$$u(x,t) = \frac{1}{2}\sum_{n=-N}^{N} A_n(t)\mathrm{e}^{\mathrm{i}nx}, \qquad \text{with} \qquad A_{-n} = A_n^*.$$

Assume $A_n \sim \epsilon^n$ (to be checked a posteriori), and keep the first three harmonics:

$$\begin{aligned}
\frac{\mathrm{d}A_1}{\mathrm{d}t} &= \sigma_1 A_1 - \mathrm{i}VA_1^* A_2 + \mathcal{O}(A_1^5),\\
\frac{\mathrm{d}A_2}{\mathrm{d}t} &= \sigma_2 A_2 - \mathrm{i}VA_1^2 + \mathcal{O}(A_1^4),\\
\frac{\mathrm{d}A_3}{\mathrm{d}t} &= \sigma_3 A_3 - 3\mathrm{i}VA_1 A_2 + \mathcal{O}(A_1^5).
\end{aligned}$$

KS: Reduction to the central manifold

Close to threshold, $\mathrm{d}/\mathrm{d}t \sim \sigma_1 \ll 1$, and $|\sigma_n| \gg \sigma_1$, so that

$$A_2 = \frac{\mathrm{i}V}{\sigma_2} A_1^2 + \mathcal{O}(A_1^4),$$
$$A_3 = \frac{3\mathrm{i}V}{\sigma_3} A_1 A_2 + \mathcal{O}(A_1^5) \sim -\frac{3V^2}{\sigma_2 \sigma_3} A_1^3.$$

$\rightarrow$ All the harmonics are 'slaved' to the fundamental.

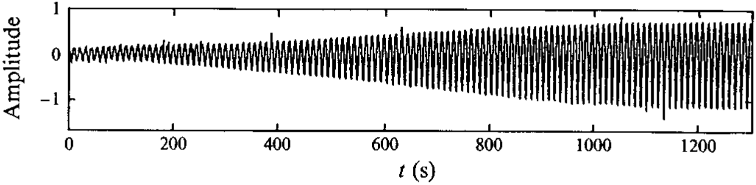
The dynamics of the fundamental is governed by the Landau equation

$$\frac{\mathrm{d}A_1}{\mathrm{d}t} = \sigma_1 A_1 - \kappa |A_1|^2 A_1 + \mathcal{O}(A_1^5), \qquad \kappa = -\frac{V^2}{\sigma_2} > 0$$

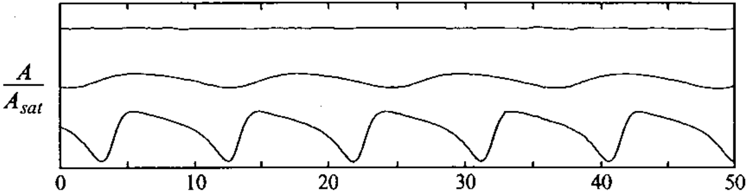
$\rightarrow$ Supercritical Hopf bifurcation at $R = 1$.

Illustration: waves at a sheared interface (Barthelet, Charru & Fabre 1995)

Two-layer Couette flow experiments in a annular channel, of mean radius $R = 0.4$ m.
The interface between the two viscous fluids becomes unstable beyond some critical upper plate velocity U: a long wave grows with $\lambda = 2\pi R$.

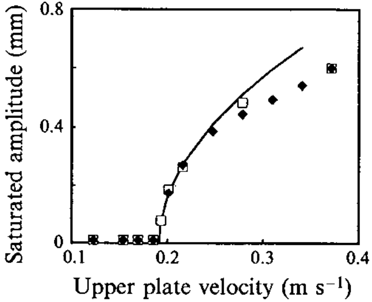


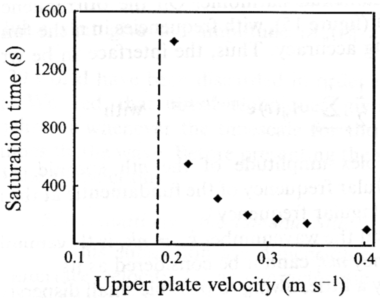
Saturated wave just below the threshold, just above, and farther:



Sheared interface (2): bifurcation diagram

Bifurcation diagram (no hysteresis), and saturation time:

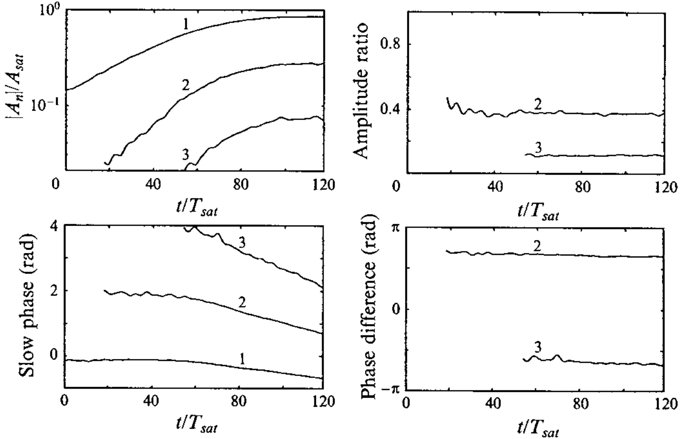




Sheared interface (3): dynamics of the harmonics

Check that $A_2 \propto A_1^2$ and $A_3 \propto A_1^3$ as predicted by the theory?

Time evolution of the harmonics $\frac{1}{2} A_n(t) \mathrm{e}^{in(k_1 x - \omega_1^0 t)} + \mathrm{c.c.}$, with amplitudes $A_n(t) = |A_n(t)| \mathrm{e}^{\mathrm{i}\phi_n(t)}$, obtained by pass-band filtering about the frequency $n\omega_1^0$. Modulus $|A_n(t)|$ and slow phases $\phi_n(t)$ obtained by Hilbert transform:



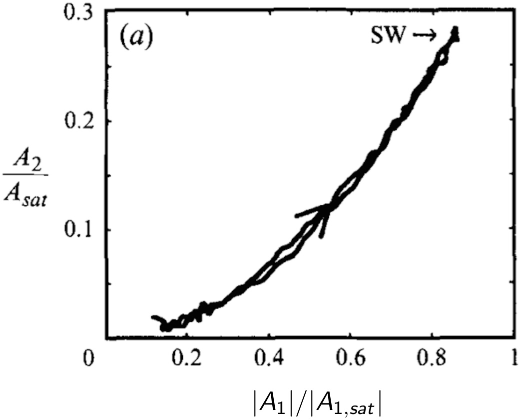
$|A_2|/|A_1|^2$

$|A_3|/|A_1|^3$

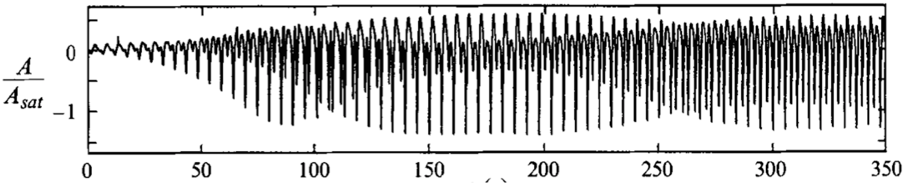
$\phi_2 - 2\phi_1$

$\phi_3 - 3\phi_1$

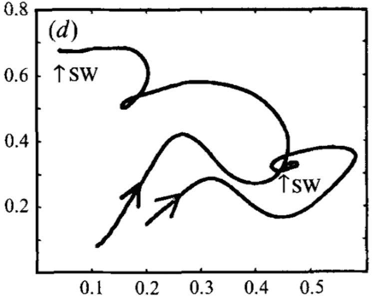
Sheared interface (4): experimental center manifold



Sheared interface (5): farther from threshold...



$|A_2|/|A_{2,sat}|$



$|A_1|/|A_{1,sat}|$

Saturated wave:

$\lambda = 2\pi R$ or $\frac{1}{2} 2\pi R$

6. Nonlinear dispersive waves

- Surface gravity waves of amplitude 'not small' are not sinusoidal
- The dispersion relation $\omega_0^2 = gk_0$ is not accurately satisfied
- How harmonics can propagate with the same velocity as the fundamental?
- What is the stability of finite amplitude waves?

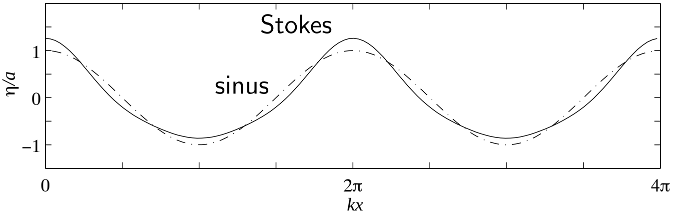
Finite amplitude gravity waves: Stokes 1847

Using a series expansion in powers of the wave slope $\epsilon = k_0 a_0$, Stokes (1847) found the profile of the free surface $\eta(x, t)$

$$\frac{\eta(x,t)}{a_0} = \frac{\epsilon}{2} + \cos\theta + \frac{\epsilon}{2}\cos 2\theta + \frac{3\epsilon^2}{8}\cos 3\theta + \mathcal{O}(\epsilon^3),$$

with phase $\theta = k_0 x - \omega t$ and frequency

$$\omega = \omega_0\left(1 + \frac{1}{2}\epsilon^2 + \mathcal{O}(\epsilon^4)\right), \qquad \omega_0^2 = g k_0.$$



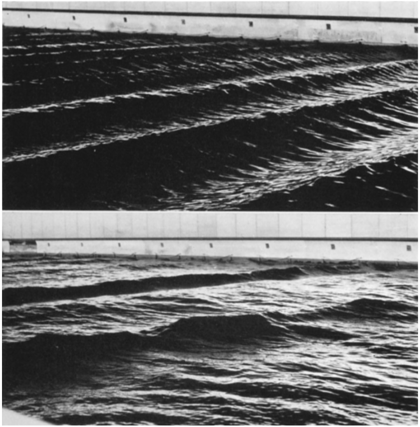
The Stokes wave is unstable

(Benjamin & Hasselman 1967)

The progressive wave train ($\lambda = 2.2$ m), degenerates into a series of wave groups, and eventually disintegrates:

near the wave-maker

60 m downstream



The Benjamin-Feir instability

(Benjamin & Feir 1967)

The instability of gravity waves is a generic instability of dispersive waves, of wave number k_0, to perturbations with nearby wave numbers $k_0 + \delta k$, now known as a side-band instability.

These perturbations grow exponentially via a resonance mechanism when

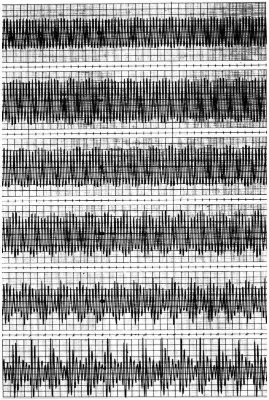
$$\frac{\delta k^2}{k_0^2} < 8(k_0 a_0)^2,$$

The two most highly amplified perturbations are those with wave numbers $k_0(1 \pm 2k_0 a_0)$, and their growth rate is

$$\sigma_{\max} = \frac{\omega_0}{2}(k_0 a_0)^2.$$

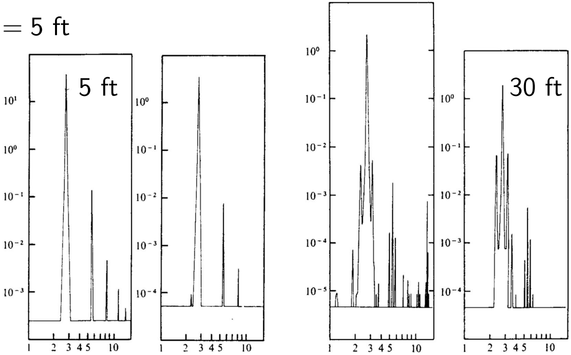
Experimental validation of the theory

(Lake & Yuen 1977)



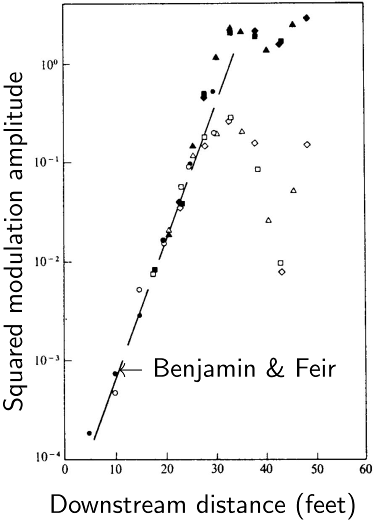
$x = 5$ ft

$x = 30$ ft

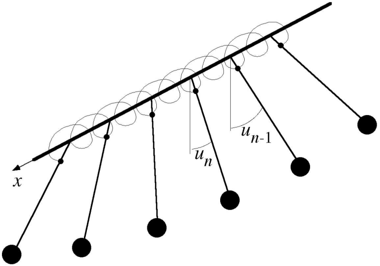


Experimental validation of the theory (2)

(Lake & Yuen 1977)



Model problem: a chain of coupled oscillators



In the long-wave limit and with appropriate choice of the time, mass and length scales, the equation of motion reduces to the nonlinear Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -V'(u), \qquad V(u) = \frac{u^2}{2} + \gamma u^4, \quad \gamma = \frac{1}{24}$$

Dispersion relation of waves with infinitesimal amplitude (no instability):

$$\omega^2 = 1 + k^2.$$

Model problem (1): nonlinear Klein-Gordon wave

Seek a traveling wave solution propagating in the x-direction ($c = \omega/k > 0$) as

$$u(x,t) = \frac{1}{2} \sum_{n=-N}^{N} \epsilon A_n(t) \mathrm{e}^{\mathrm{i}(k_n x - \omega_n t)},$$

The time scale of the nonlinear interactions is of order ϵ^{-2}. Introducing the slow time scale $T = \epsilon^2 t$, we obtain the amplitude equation for the nth mode:

$$\frac{\mathrm{d}A_n}{\mathrm{d}T} = -\frac{\mathrm{i}\gamma}{2\omega_n} \sum_{k_p+k_q+k_r=k_n} A_p A_q A_r \mathrm{e}^{\mathrm{i}(\omega_n-\omega_p-\omega_q-\omega_r)T/\epsilon^2}.$$

This interaction leads to remarkable solutions, in particular, when the frequencies satisfy the very special *resonance condition*

$$\omega_p + \omega_q + \omega_r = \omega_n.$$

Model problem (2): nonlinear Klein-Gordon wave

Let us consider the resonant interaction of a wave of wave number k_0 with itself (self-interaction). The summation runs over 2^3 triads $(\pm k_0, \pm k_0, \pm k_0)$, only three satisfy the resonance condition: $(k_0, k_0, -k_0)$, $(k_0, -k_0, k_0)$, and $(-k_0, k_0, k_0)$.

The amplitude equation for A_0 then reduces to

$$\frac{\mathrm{d}A_0}{\mathrm{d}T} = -\mathrm{i}\beta A_0^2 A_0^*, \qquad \beta = \frac{3\gamma}{2\omega_0}, \quad \omega_0 = \sqrt{1 + k_0^2}.$$

with solution $A_0 = a_0 \mathrm{e}^{-\mathrm{i}\beta a_0^2 T}$, $a_0 = \mathcal{O}(1)$ real.

Returning to the original angular variable

$$u(x,t) = \epsilon a_0 \cos(k_0 x - \omega t) + \mathcal{O}(\epsilon^3), \qquad \omega = \omega_0 + \beta(\epsilon a_0)^2$$

The frequency and speed of the wave are modified by the self-interaction due to the cubic nonlinearity: they depend on the amplitude.
The frequency correction is the same as that of a Duffing oscillator.

Model problem (3): stability of the nonlinear wave

Consider the effect of a perturbation of the monochromatic wave in the form of two waves with wave numbers close to k_0:

$$k_0 \pm \epsilon K \text{ with } K = \mathcal{O}(1), \quad \text{frequencies } \omega_\pm, \quad \text{amplitudes } |A_\pm| \ll |A_0|.$$

Keeping only the dominant terms, the amplitude equations reduce to

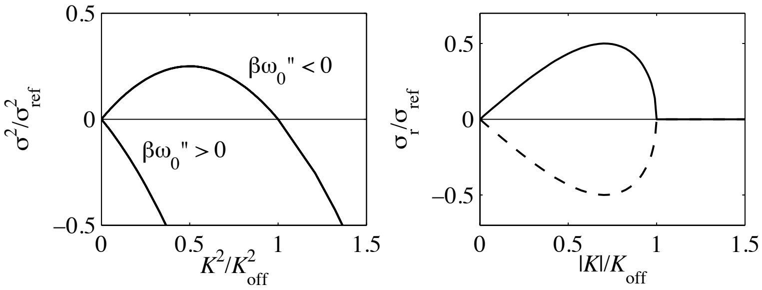
$$\begin{aligned} \frac{\mathrm{d}A_-}{\mathrm{d}T} &= -\mathrm{i}\beta a_0^2 \left(2A_- + A_+^* \mathrm{e}^{i(\Omega - 2\beta a_0^2)T}\right) \\ \frac{\mathrm{d}A_+}{\mathrm{d}T} &= -\mathrm{i}\beta a_0^2 \left(2A_+ + A_-^* \mathrm{e}^{i(\Omega - 2\beta a_0^2)T}\right). \end{aligned}$$

$$\text{with } \Omega = \frac{1}{\epsilon^2}(\omega_+ + \omega_- - 2\omega_0) \approx \omega_0'' K^2 = \mathcal{O}(1), \quad \omega_0'' = \frac{\partial^2 \omega}{\partial k^2}(k_0) = \omega_0^{-3}.$$

This system can be made autonomous by a rotation of $A_\pm$ in the complex plane, and has nontrivial solutions $\propto \mathrm{e}^{\sigma T}$ if (dispersion relation)

$$\sigma^2 + \beta\omega_0'' a_0^2 K^2 + \frac{\omega_0''^2}{4} K^4 = 0.$$

Model problem (4): stability of the nonlinear wave



A necessary condition for instability ($\sigma_r \neq 0$) is $\beta\omega_0'' < 0$.

Then, the wave is unstable to side-band perturbations of wave numbers

$$k = k_0 \pm \epsilon K \text{ with } K < K_{\text{off}} = 2a_0\sqrt{-\beta/\omega_0''}.$$

The two most amplified wave numbers are

$$k_{\max} = k_0 \pm \frac{1}{\sqrt{2}}\epsilon K_{\text{off}}.$$

The Stokes wave instability revisited

The instability condition $\beta\omega_0'' < 0$ established for the Klein–Gordon wave is actually general: it is valid for any dispersive nonlinear wave, with β the coefficient of the nonlinear correction $\beta(\epsilon a_0)^2$ to the frequency.

For example, for a gravity wave we obtain from the dispersion relation for finite-amplitude waves found by Stokes:

$$\omega_0'' = -\frac{\omega_0}{4k_0^2}, \qquad \beta = \frac{1}{2}\omega_0 k_0^2.$$

The instability condition is then

$$(\epsilon K)^2 < 8k_0^4(\epsilon a_0)^2$$

identical to that obtained by Benjamin and Feir (1967) in their solution of the hydrodynamical problem! This is no accident, as the instability results from a competition between the linear dispersion and the nonlinearity, the effect of the latter being contained entirely in the nonlinear correction of the wave frequency.

Alternative analysis: dynamics of a wave packet

(Benney & Newell 1967; Stuart & DiPrima 1978)

A wave packet centered on the wave number k_0 propagating in the direction of increasing x can be represented as the Fourier integral

$$u(x,t) = \frac{1}{2}A(x,t)\mathrm{e}^{i(k_0 x - \omega_0 t)} + \mathrm{c.c.}$$

where $\omega_0 = \omega(k_0)$ (real) and the *envelope* $A(x,t)$ of the wave packet is defined as

$$A(x,t) = \int_0^{+\infty} \hat{u}(k)\mathrm{e}^{\mathrm{i}(k-k_0)x - \mathrm{i}(\omega(k)-\omega_0)t} dk.$$

Expand $\omega(k)$ in Taylor series about k_0 and truncate at second order:

$$\omega - \omega_0 = c_\mathrm{g}(k - k_0) + \frac{\omega_0''}{2}(k-k_0)^2 \qquad \text{with } c_\mathrm{g} = \frac{\partial \omega}{\partial k}(k_0), \quad \omega_0'' = \frac{\partial^2 \omega}{\partial k^2}(k_0).$$

We recognize the general solution of the envelope equation

$$\mathrm{i}\,\frac{\partial A}{\partial t} = -\mathrm{i}c_\mathrm{g}\,\frac{\partial A}{\partial x} + \alpha\,\frac{\partial^2 A}{\partial x^2}, \qquad \alpha = \frac{1}{2}\omega_0''.$$

Nonlinear dynamics: the nonlinear Schrödinger equation

According to the above linear envelope equation, the width of the wave packet increases linearly with time due to dispersion, while its amplitude decreases as $1/\sqrt{t}$. Nonlinearity may counteract dispersion.

For problems invariant under translations $x \to x + \xi$ and $t \to t + \tau$, the nonlinear envelope equation must be invariant under the transformation $A \to A\mathrm{e}^{\mathrm{i}\phi}$. Hence the nonlinear Schrödinger (NLS) equation:

$$\mathrm{i}\,\frac{\partial A}{\partial t} = -\mathrm{i}c_{\mathrm{g}}\,\frac{\partial A}{\partial x} + \alpha\,\frac{\partial^2 A}{\partial x^2} - \beta\,|A|^2 A.$$

If the problem is invariant under reflections $x \to -x$ and $t \to -t$, β is real.

For the coupled pendulum problem, a multiple scale analysis shows $\beta = 3\gamma/2\omega_0$.

Stability of a quasi-monochromatic wave (1)

The nonlinear Schrödinger equation admits the spatially uniform solution

$$A_0 = a_0 \mathrm{e}^{\mathrm{i}(\Omega t + \Phi)}, \qquad a_0 = |A_0| \text{ real}, \qquad \Omega = \beta a_0^2,$$

which corresponds to the unmodulated traveling wave

$$u(x,t) = a_0 \cos(k_0 x - \omega t + \Phi), \qquad \omega = \omega_0 + \beta a_0^2,$$

Stability of a quasi-monochromatic wave (2)

Perturb A_0 as

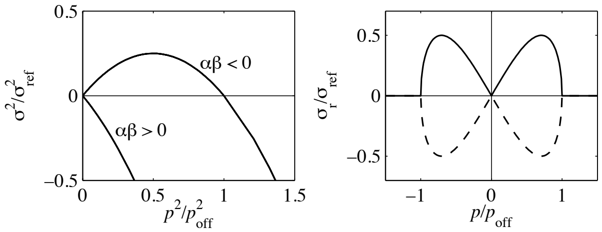
$$A(x,t) = (a_0 + a(x,t))\mathrm{e}^{\mathrm{i}(\Omega t + \Phi + \varphi(x,t))}$$

substitute in the NLS, linearize and separate the real and imaginary parts:

$$\partial_t a = \alpha a_0 \, \partial_{xx} \varphi,$$
$$\partial_t \varphi = 2\beta a_0 a - (\alpha / a_0)\, \partial_{xx} a.$$

This linear system admits solutions of the form $\mathrm{e}^{\sigma t - \mathrm{i} p x}$, with (dispersion relation):

$$\sigma^2 + 2\alpha\beta a_0^2 \, p^2 + \alpha^2 \, p^4 = 0.$$



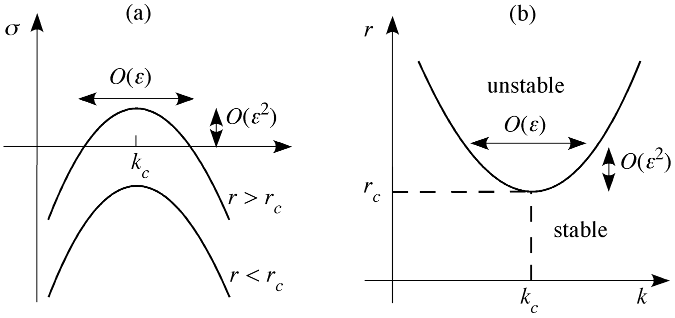
$\leftarrow \alpha\beta < 0$

The Benjamin-Feir (side-band) instability is exactly recovered.

7. Nonlinear dynamics of dissipative systems

- What happens when the size of the system is large compared with the wavelength of an unstable mode for $R \simeq R_c$?
- We first consider systems with the translational and reflectional symmetries
- We then consider propagating waves (no reflectional symmetry)

Linear analysis, for $\omega = \partial\omega/\partial k = 0$ at $(k, R) = (k_c, R_c)$



Close to threshold ($\epsilon^2 = r - r_c \ll 1$ with $r = R/R_c$), expand the growth rate:

$$\tau_{\mathrm{c}}\, \sigma(k, r) = (r - r_{\mathrm{c}}) - \xi_{\mathrm{c}}^2 (k - k_{\mathrm{c}})^2 + \dots ,$$

where τ_{c} and ξ_{c} are characteristic time and length scales defined as

$$\frac{1}{\tau_{\mathrm{c}}} = \frac{\partial \sigma}{\partial r}, \qquad \frac{\xi_{\mathrm{c}}^2}{\tau_{\mathrm{c}}} = -\frac{1}{2} \frac{\partial^2 \sigma}{\partial k^2}.$$

Dynamics of a wave packet

The perturbation of the base state can be written as

$$u(x,t) = \frac{1}{2}\mathcal{A}(x,t)\mathrm{e}^{\mathrm{i}k_\mathrm{c}x} + \mathrm{c.c.},$$

where the envelope $\mathcal{A}(x,t)$ of the wave packet is defined as

$$\mathcal{A}(x,t) = \int_0^{+\infty} \hat{u}(k)\mathrm{e}^{\mathrm{i}(k-k_\mathrm{c})x+\sigma(k)t}\mathrm{d}k.$$

Replacing $\sigma(k)$ by its Taylor series, we recognize the general solution of the envelope equation

$$\tau_\mathrm{c}\frac{\partial \mathcal{A}}{\partial t} = (r-r_\mathrm{c})\mathcal{A} + \xi_\mathrm{c}^2\frac{\partial^2 \mathcal{A}}{\partial x^2}.$$

For systems invariant under space and time translation, the weakly noninear dynamics is governed by the Ginzburg–Landau envelope equation with real κ:

$$\tau_\mathrm{c}\frac{\partial \mathcal{A}}{\partial t} = (r-r_\mathrm{c})\mathcal{A} + \xi_\mathrm{c}^2\frac{\partial^2 \mathcal{A}}{\partial x^2} - \kappa|\mathcal{A}|^2\mathcal{A},$$

Saturated pattern, and linear stability (1)

Periodic pattern. For $\kappa > 0$, the Ginzburg–Landau equation possesses a continuous family of uniform, stationary solutions

$$U_0(x,t) = u_0 \cos(k_0 x + \Phi)$$

of amplitude u_0 and wave number k_0 defined as

$$u_0 = \sqrt{r - r_\mathrm{c}}\sqrt{\frac{1-q_0^2}{\kappa}}, \qquad k_0 = k_\mathrm{c} + \epsilon q_0/\xi_\mathrm{c}, \qquad -1 \le q_0 \le 1.$$

Stability. Perturb the amplitude as $a_0 + \tilde{a}(X,T)$ and the phase as $\Phi + \varphi(X,T)$, linearize, and find

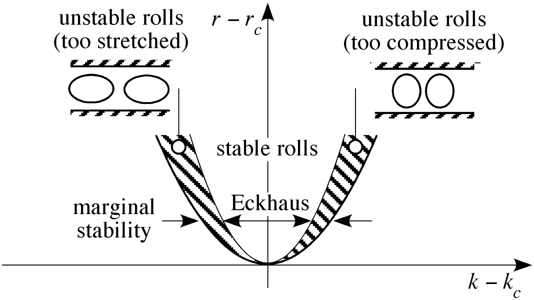
$$\partial_T \tilde{a} = -2a_0^2\,\tilde{a} + \partial_{XX}\tilde{a} - 2a_0 q_0\,\partial_X\varphi,$$
$$\partial_T\varphi = -\frac{2q_0}{a_0}\,\partial_X\tilde{a} + \partial_{XX}\varphi.$$

This system admits solutions $\propto \mathrm{e}^{\mathrm{i}pX+\sigma T}$, with (dispersion relation)

$$\sigma_\pm = -(a_0^2 + p^2) \pm \sqrt{a_0^4 + 4q_0^2 p^2}.$$

Saturated pattern, and linear stability (2)

The amplitude mode is stable ($\sigma_- < 0$), and slaved to the phase mode which is unstable ($\sigma_+ > 0$) for $q_0^2 > 1/3$.

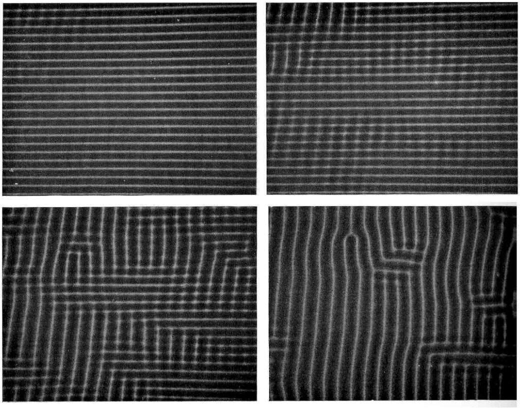


It can be shown that the instability is subcritical: no saturation mechanism.

Illustration: Rayleigh-Bénard convection (1)

The roll pattern, initially 'compressed' (thermal impression technique), relaxes to larger wavelength through a 'cross-roll' instability.

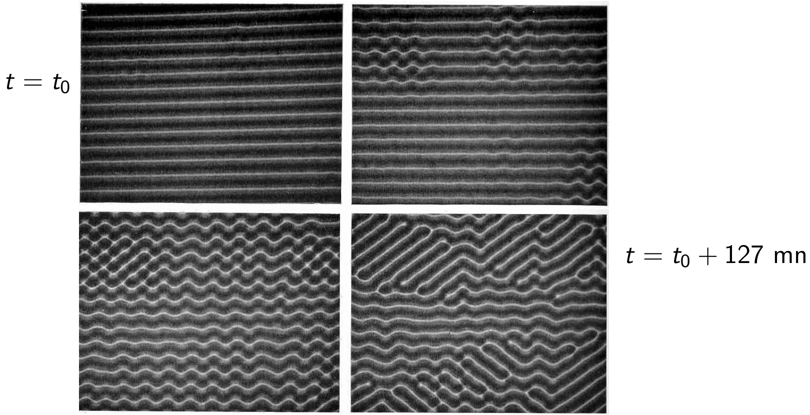
$t = t_0$



$t = t_0 + 52$ mn

Illustration: Rayleigh-Bénard convection (2)

The roll pattern, initially ‘stretched’ (thermal impression technique), relaxes to smaller wavelength through a ‘zig-zag’ instability.



Travelling dissipative waves

Consider a wave packet near the instability threshold ($\sigma = 0$ and $\omega = \omega_c$ at the critical point (k_c, R_c)), expand the dispersion relation, take the inverse Fourier transform, add the dominant nonlinear term $|A|^2 A$.

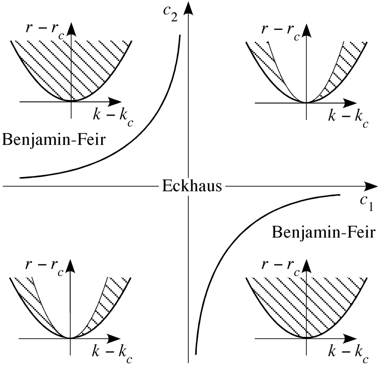
We obtain the complex Ginzburg-Landau (CGL) equation

$$\tau_{\mathrm{c}} \left(\frac{\partial \mathcal{A}}{\partial t} + c_{\mathrm{g}} \frac{\partial \mathcal{A}}{\partial x} \right) = (r - r_{\mathrm{c}})\mathcal{A} + (\xi_{\mathrm{c}}^2 + \frac{\mathrm{i}\, \tau_{\mathrm{c}} \omega_{\mathrm{c}}''}{2}) \frac{\partial^2 \mathcal{A}}{\partial x^2} - \kappa |\mathcal{A}|^2 \mathcal{A},$$

Finite-amplitude travelling waves, and their stability

The CGL equation possesses a continuous family of travelling wave solutions. Instability corresponds to negative diffusion in the equation of the phase perturbation:

$$D(q_0) = 1 + c_1 c_2 - 2q_0^2\,(1 + c_2^2)/(1 - q_0^2), \qquad -1 \leq q_0 \leq 1.$$



The Benjamin-Feir and Eckhaus instability are unified (Stuart & DiPrima 1978).